The Satake Isomorphism

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Chapter 0

We are going to study:

$\underline{G}$ a **connected reductive split algebraic group** over a local field F with uniformizer π

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Ex.: $\underline{G} = GL_n, SL_n, SO_n, Sp_{2n}, \dots$

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An (affine) **algebraic group** is...

- (Classical viewpoint) An algebraic variety $V \subseteq F^n$ which is also a group (e.g. $\mathbb{G}_a := (F, +, 0)$)
- (Modern viewpoint) A representable functor

$$\underline{G} \colon F\text{–}\mathbf{Alg} \to \mathbf{Grp}$$

(e.g. $\mathbb{G}_m : A \mapsto A^\times$)

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Each such group is irreducible, therefore **connected**

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An algebraic group is **reductive** if its only normal unipotent connected subgroup is the trivial one $\{e\}$

- An algebraic group is unipotent if it is isomorphic to a subgroup of

$$U_n = \{\text{upper triangular matrices with diagonal 1}\}$$

  (e.g. $\mathbb{G}_a = U_2$)

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A reductive algebraic group is **split** if its maximal torus is split

- An algebraic group is a torus if it is isomorphic to $(\mathbb{G}_m)^n$ over $\overline{F}$ for some n
  (e.g. SO_2 over $\mathbb{Q}$)
- A torus is split if it is already isomorphic to $(\mathbb{G}_m)^n$
  (e.g. the group of invertible diagonal matrices)

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Fix $\underline{T} \subset \underline{B} \subset \underline{G}$ a maximal torus and a Borel subgroup containing it and let $\underline{N}$ be the unipotent radical of $\underline{B}$ (then $\underline{B} = \underline{N} \rtimes \underline{T}$)

(e.g. $G = \mathrm{GL}_n$, $B =$ upper triangular matrices, $T =$ invertible diagonal matrices and $N = U_n$)

Introduction

We would like to study well-behaved representations of $G := \underline{G}(F)$

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Problem: $\mathcal{H}(G)$ does not have unity

Solution:

$$\mathcal{H}(G) = \bigcup_{\substack{K<G \\ \text{compact open}}} \mathcal{H}(G, K)$$

The Spherical Hecke Algebra

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$$\mathcal{H}(G, K) = \mathbb{C}_c[K \backslash G / K]$$

If $T = \underline{T}(F)$ and $K_T = \underline{T}(\mathcal{O}_F) = T \cap K$, then we will describe the spherical Hecke algebra using

$$\mathcal{H}(T, K_T)$$

(Co)characters and roots

For an algebraic group $\underline{G}$, we define

$$X^\bullet(\underline{G}) = \mathsf{Hom}(\underline{G}, \mathbb{G}_m)$$
$$X_\bullet(\underline{G}) = \mathsf{Hom}(\mathbb{G}_m, \underline{G})$$

with a pairing

$$X^\bullet(\underline{G}) \times X_\bullet(\underline{G}) \to \mathsf{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$$
$$(\alpha, \lambda) \mapsto \alpha \circ \lambda \longrightarrow \langle \alpha, \lambda \rangle$$

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An element $\lambda \in X_\bullet(T) \coloneqq X_\bullet(\underline{T})(F)$ is determined by $\lambda(\pi)$.

On the other hand, $c_\lambda \coloneqq \mathbb{1}_{K_T \lambda(\pi) K_T}$ generates $\mathcal{H}(T, K_T)$.

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There is an isomorphism

$$\mathcal{H}(T, K_T) \to \mathbb{C}[X_\bullet(T)]$$
$$c_\lambda \mapsto [\lambda]$$

Positive roots and Weyl chamber

Since T is diagonalisable, the adjoint representation may be decomposed as

$$\mathfrak{g} = \bigoplus_{\alpha \in X^\bullet(T)} V_\alpha = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} V_\alpha$$

and analogously

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} V_\alpha$$

The positive roots Φ^+ determine a Weyl chamber

$$P^+ = \{\lambda \in X_\bullet(T) \mid \langle \alpha, \lambda \rangle \geq 0 \quad \forall \alpha \in \Phi^+\}$$

Satake Isomorphism (version 1): There is an isomorphism

$$\mathcal{H}(G,K) \cong \mathcal{H}(T,K_T)^W$$

where $W = W(G,T) = N_G(T)/T$ is the Weyl group and acts by conjugation

The Satake Isomorphism

Satake Isomorphism (version 2): There is an isomorphism

$$\mathcal{H}(G, K) \cong \mathbb{C}[X_\bullet(T)]^W$$

where $W = W(G, T) = N_G(T)/T$ is the Weyl group and acts by conjugation

Definition of the isomorphism

Let dn be the left Haar measure on $N := \underline{N}(F)$ and $\delta\colon B \to \mathbb{R}_{>0}$ be the modular function of B defined by

$$d(bnb^{-1}) = d(nb^{-1}) = \delta(b)\, dn$$

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As we have seen in class, the map is going to be

$$S\colon \mathcal{H}(G, K) \to \mathcal{H}(T, K_T)$$

defined by

$$Sf(t) := \delta(t)^{1/2} \int_N f(tn)\,dn = \delta(t)^{-1/2} \int_N f(nt)\,dn$$

Algebra Homomorphism

We may see this morphism as a composition of

$$\mathcal{H}(G, K) \xrightarrow{\alpha} \mathcal{H}(B) \xrightarrow{\beta} \mathcal{H}(T) \xrightarrow{\gamma} \mathcal{H}(T)$$

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$$\mathcal{H}(G, K) \xrightarrow{\alpha} \mathcal{H}(B) \xrightarrow{\beta} \mathcal{H}(T) \xrightarrow{\gamma} \mathcal{H}(T)$$

where α is the restriction and it is an algebra homomorphism as a consequence of the **Iwasawa Decomposition** ($G = BK$)

$$\int_G u(g)\, dg = \int_B \int_K u(bk)\, dk\, db$$

Algebra Homomorphism

We may see this morphism as a composition of

$$\mathcal{H}(G,K) \xrightarrow{\alpha} \mathcal{H}(B) \xrightarrow{\beta} \mathcal{H}(T) \xrightarrow{\gamma} \mathcal{H}(T)$$

β defined by $(\beta f)(t) = \int_N f(tn)\,dn$ is an algebra homomorphism
We define the left Haar measure in B by

$$\int_B f(b)\,db := \int_T \int_N f(tn)\,dn\,dt$$

Algebra Homomorphism

We may see this morphism as a composition of

$$\mathcal{H}(G,K) \xrightarrow{\alpha} \mathcal{H}(B) \xrightarrow{\beta} \mathcal{H}(T) \xrightarrow{\gamma} \mathcal{H}(T)$$

and γ is the function $(\gamma f)(t) = \delta(t)^{1/2} f(t)$ which is also an algebra homomorphism

Invariance of the image

Remember that $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}$ is a direct sum of T-representations.

Let

$$\Delta(t) := |\det(\mathrm{Ad}_{\mathfrak{n}}(t) - \mathbb{1}_{\mathfrak{n}})|_F$$

Lemma. If $\Delta(t) \neq 0$, then

$$\int_N f(tn)\, dn = \Delta(T) \int_{G/T} f(gtg^{-1})\, d\overline{g}$$

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Lemma. If $\Delta(t) \neq 0$, then

$$\int_N f(tn)\, dn = \Delta(T) \int_{G/T} f(gtg^{-1})\, d\overline{g}$$

Therefore

$$Sf(t) = D(t) \int_{G/T} f(gtg^{-1}) d\overline{g}$$

for $D(t) := \Delta(t)\delta(t)^{-1/2}$ (if $\Delta(t) \neq 0$)

Invariance of the image

After some computations, we conclude

$$D(t) = \left|\det(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{t}}(t) - \mathbb{1}_{\mathfrak{g}/\mathfrak{t}})\right|_F^{1/2}$$

therefore

$$D(wtw^{-1}) = D(t) \quad \forall w \in N_G(T)$$

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therefore

$$D(wtw^{-1}) = D(t) \quad \forall w \in N_G(T)$$

Since $N_G(T) \cap K$ is compact, its action preserves the Haar measure on G and T, hence on the quotient G/T, which implies

$$\int_{G/K} f(g(xtx^{-1})g^{-1})\, d\overline{g} = \int_{G/K} f(gtg^{-1}) d\overline{g}$$

for $x \in N_G(T) \cap K$

Isomorphism

By **Cartan decomposition** ($G = KTK$), $\{c_\lambda\}_{\lambda \in P^+}$ is a basis of $\mathcal{H}(G, K)$.

Moreover the elements

$$\{\lambda\} := \frac{1}{\mathrm{Stab}(\lambda)} \sum_{w \in W} [w \cdot \lambda] \in \mathbb{C}[X_\bullet(T)]^W$$

form a basis for $\lambda \in P^+$

Let $[c(\lambda, \lambda')]$ be the matrix defined by

$$Sc_\lambda = \sum_{\lambda' \in P^+} c(\lambda, \lambda')\{\lambda'\}$$

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Let $[c(\lambda, \lambda')]$ be the matrix defined by

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Idea: We'll prove $[c(\lambda, \lambda')]$ is upper triangular with non-zero elements in the diagonal

Isomorphism

Let $2\rho = \sum_{\Phi^+} \alpha$ and define

$$\lambda > \mu \iff \langle \rho, \lambda - \mu \rangle > 0$$

We then "complete" this partial order so that it is linear

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Computing the integral, we get

$$c(\lambda, \lambda') = Sc_\lambda(\lambda'(\pi)) = \delta(\lambda'(\pi))^{1/2} \cdot \mu(K\lambda(\pi)K \cap N\lambda'(\pi)K)$$

In particular

$$c(\lambda, \lambda) \geq \delta(\lambda'(\pi))^{1/2} \cdot \mu(\lambda(\pi)K) \geq \delta(\lambda'(\pi))^{-1/2}$$

■

The Satake Yoga: There is a representation theoretic description of automorphic forms.

Satake Isomorphism (version 3): There is an isomorphism

$$\mathcal{H}(G, K) \cong R({}^{L}G) \otimes_{\mathbb{Z}} \mathbb{C}$$