Sorbonne Université

Master Thesis

The Geometric Satake Equivalence

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Introduction

The Satake isomorphism is a now classical result of the Langlands program which builds a bridge between Analysis and Representation Theory. It says that if G is a split reductive group over a non-archimedean local field F, then there is an isomorphism between the spherical Hecke algebra[1] and the complex representation ring of the Langlands dual[2] of G:

$$\mathcal{H}(G, K) \cong \mathcal{R}(G^\vee).$$

Grothendieck noticed that some functions on algebraic varieties over finite fields $\mathbb{F}_q$ could be seen as "shadows" of complexes of ℓ-adic sheaves, by what is now called the *fonctions-faisceaux* dictionary. Precisely, fixed an algebraic variety X over a finite field $\mathbb{F}_q$, one can associate to each complex of ℓ-adic sheaves $\mathscr{F}$ the function

$$x \mapsto \chi_{\mathscr{F}}(x) := \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{tr}(\mathrm{Fr}, \mathscr{H}_x^i(\mathscr{F})),$$

defined on $X(\mathbb{F}_q)$. Although conceptually more complicated, sheaves are much more versatile. For example, the upper star specializes to the pullback, the tensor product specializes to multiplication and the lower shriek specializes to integration along the fibers.

The publication of *Faisceaux Pervers* [5] and further work by Deligne opened the doors for this new idea, being, for example, an important tool used by Laumon on his simplication of the Weil II.

The idea of using Perverse sheaves on affine Grassmannians was inspired by the works of Lusztig [17, 18] on irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ and of Drinfeld [11] on automorphic forms on GL_2. These works were Ginzburg's main motivation to turn the Satake isomorphism into a geometric statement (while also losing its p-adic shell). In 2007, a more general version was proved by Mirković and Vilonen, which we state below.

Let G be a complex connected reductive algebraic group, let k be a Noetherian commutative ring with finite global dimension, and let Gr_G be the infinite-dimensional complex variety defined as $G(\mathbb{C}(\!(t)\!))/G(\mathbb{C}[\![t]\!])$. Then there is an isomorphism of Tannakian categories between the $G(\mathbb{C}[\![t]\!])$-equivariant perverse sheaves on Gr_G and the representations of the Langlands dual group:

$$\mathrm{P}_{L^+G}(\mathrm{Gr}_G, k) \cong \mathsf{Rep}_k(G_k^\vee).$$

This gives both a canonical construction of the Langlands dual (or, more generally, the Chevalley scheme) and another geometric tool to study Representation Theory.

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[1]The spherical Hecke algebra is the algebra of $G(\mathcal{O}_F)$-bi-invariant functions $f\colon G(F) \to \mathbb{C}$ with compact support.

[2]The Langlands dual group of G is the unique k-split reductive group $G_k^\vee$ over k whose root datum is dual to the one coming from G. In spite of having an indirect definition, it is not hard to compute the Langlands dual for some simple algebaic groups. For example, if $G = \mathrm{GL}_n$, then $G^\vee = \mathrm{GL}_n$ and if $G = \mathrm{SL}_n$, then $G^\vee = \mathrm{PGL}_n$.

Although this statement is not related to the original geometric Satake, it is possible to find a similar proof [21, 14.1] working with the étale topology. This allows us to consider G over an arbitrary algebraically closed field K of characteristic p and $k = \overline{\mathbb{Q}}_\ell$ for $\ell \neq p$.

It is possible to recover the Satake isomorphism on positive characteristic with this new statement [26, Section 5.6], but it involves a subtle argument relating the Satake category $\mathrm{P}_{L^+G}(\mathrm{Gr}_{G_{\overline{\mathbb{F}_q}}}, \overline{\mathbb{Q}_\ell})$ with a subcategory of $\mathrm{P}_{L^+G}(\mathrm{Gr}_G, \overline{\mathbb{Q}_\ell})$ for a reductive group G over $\mathbb{F}_q$.

This master thesis only proves a special case of the Geometric Satake equivalence. The final theorem we are going to prove is the following result.

Theorem. *Let G be a complex reductive algebraic group and k a field of characteristic zero, then there is an equivalence of Tannakian categories*[3]

$$\mathrm{P}_{L^+G}(\mathrm{Gr}_G, k) \cong \mathsf{Rep}_k(G_k^\vee).$$

The chapters are organized in the following way. The first chapter gives the definition of a (neutral) Tannakian category and states the main theorem of the Tannakian formalism. It also gives statements relating the structure of the category $\mathsf{Rep}_k(G)$ with the geometry of the group scheme G, which are useful to conclude that the group found by the Tannakian formalism is, in fact, the Langlands dual group.

The second chapter gives three equivalent definitions of the affine Grassmannian, from the most elementary and restricted to GL_n to a more technical using étale quotients. Moreover, it gives the decomposition of the affine Grassmannian into Schubert cells and the decomposition into semi-infinite orbits. The former being the heart of most arguments involving the affine Grassmannian, and the second one is an important idea of Mirković-Vilonen to give the correct grading to the total cohomology functor.

The third chapter is an overview of the main definitions and results on constructible sheaves, t-structures and perverse sheaves. These are all prerequisites to study the geometric Satake equivalence, and some more pre-requisites are given in the appendices on Algebraic Groups and Ind-schemes.

Chapter 4 studies the Satake category $\mathrm{P}_{L^+G}(\mathrm{Gr}_G, k)$ by itself. First of all, we prove it is semisimple (a property particular to zero characteristic) and then we define a convolution. Some heavy work is necessary to define the commutativity constraint, and the Beilinson-Drinfeld Grassmannian is also defined.

Chapter 5 studies the fiber functor. In spite of being the total cohomology, the fiber functor has a more structured definition as direct sum of local cohomologies on the semi-infinite orbits, and studying this requires some dimension estimates, which are important in this chapter.

Finally, chapter 6 identifies the group given by the Tannakian formalism as the Langlands dual. First of all, we prove this group is a reductive connected algebraic group. Then we study its maximal torus and its root datum to complete the proof.

[3] As with the Satake isomorphism, the interesting part of the statement is the monoidal/multiplicative part. In fact, is not hard to prove the simple objects are indexed by the same set $X_\bullet(T)^+$ and therefore they are easily proven to be isomorphic as Abelian categories.

Notation

Unless otherwise stated, k is a field of characteristic 0 and A denotes a k-algebra. When talking about Tannakian Formalism we allow A to be non-commutative. In this case, we call A-mod the category of finite-dimensional A-modules, and for a k-coalgebra B, we denote by B-comod the category of finite-dimensional B-comodules. If G is a group scheme, then $\mathsf{Rep}_k(G)$ is the category of finite-dimensional representations of G, which is isomorphic to $\mathcal{O}_G(G)$-comod.

The category of affine schemes over k is denoted by AffSch_k and a k-functor is simply a functor $X\colon \mathsf{AffSch}_k^{op} \to \mathsf{Set}$, which is sometimes more easily described as a functor $X\colon k\text{-}\mathsf{Alg} \to \mathsf{Set}$.

For a k-algebra A, we define $\mathbb{D}_A := \mathrm{Spec}(A[\![t]\!])$ and $\mathbb{D}_A^\times := \mathrm{Spec}(A(\!(t)\!))$.

If X is a curve over k and $x \in X$ is a closed point, then $\mathcal{O}_x := \widehat{\mathcal{O}_{X,x}}$ is the completion of the local ring at x. A choice of coordinates at x fixes an isomorphism $\mathcal{O}_x \cong k[\![t]\!]$. We also define $\mathcal{K}_x := \mathrm{Frac}(\mathcal{O}_x)$, which, under the previous isomorphism gives $\mathcal{K}_x \cong k(\!(t)\!)$.

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In memory of Cláudio José Arruda de Souza Leão

Chapter 1

Tannakian Formalism

The Tannakian Formalism gives the conditions for a Abelian category $\mathcal{A}$ be isomorphic to $\mathsf{Rep}_k(G)$ for some affine group scheme G. It allows us to use representation theory to study Abelian categories appearing in all of Mathematics.

Moreover, some properties of G may be verified using the category $\mathsf{Rep}_k(G)$. For example, G is commutative if and only if the irreducible representations are 1-dimensional. This is similar to what is done in representation of algeras: if A is a k-algebra then a monoidal structure on A-mod corresponds to *comultiplication*, a dual structure corresponds to *counit* and *antipode*, etc.

In the first section, we are going to study give a condition for an Abelian category be isomorphic to A-mod for some finite-dimensional k-algebra A. In the following chapter, we generalize this and give conditions for an Abelian category be isomorphic to B-comod for some coalgebra B (since A-$\mathsf{mod} \cong A^\vee$-$\mathsf{comod}$, this is, indeed, more general). Finally, in the last two sections, we give conditions to ensure the Abelian category is isomorphic to $\mathsf{Rep}_k(G) \cong \mathcal{O}_G(G)$-$\mathsf{comod}$ for some affine group scheme G, and study properties of this group scheme through the eyes of the representation category.

1.1 Tannakian Formalism and Algebras

The first question we wish to answer is: "How to recover a k-algebra A from the category A-mod?"

The most naive way to recover A is the following.

Proposition 1. *Let A be a k-algebra and V a finite-dimensional k-vector space. If $\alpha\colon A \to \operatorname{End}(V)$ is faithful, then V has an A-module structure and*

$$A \cong \{f \in \operatorname{End}(V) \mid \forall n \geq 1\ \forall W \leq V^{\oplus n}\ A\text{-submodule},\ f^{\oplus n}(W) \subseteq W\}.$$

Proof. By definition of the module structure, if f is in $\operatorname{im}\alpha$, then for any positive integer $n \geq 1$ and any A-submodule $W \leq V^{\oplus n}$, we have $f^{\oplus n}(W) \subseteq W$.

On the other hand, let $n = \dim V$ and $\{e_1, e_2, \ldots, e_n\}$ be a basis of V. Moreover, let W be the A-submodule of $V^{\oplus n}$ generated by $(e_1, e_2, \ldots, e_n)$. Since $f^{\oplus n}(W) \subset W$, this means there exists an $a \in A$ such that

$$f(e_1, e_2, \ldots, e_n) = a \cdot (e_1, e_2, \ldots, e_n).$$

Which means $f(e_i) = a \cdot e_i$ for each e_i, therefore f is in $\operatorname{im}\alpha$. Since the map is faithful, then $A \cong \operatorname{im}\alpha$, as desired. □

Although concrete, this does not allows us to recover uniformly an algebra A from its category of (finite-dimensional) modules. In fact, one must find a faithful module.

We call an Abelian category $\mathcal{A}$ k-linear if the hom-sets are not only abelian groups, but finite-dimensional k-vector spaces[1]. For an Abelian category $\mathcal{A}$, and an object $X \in \mathcal{A}$, we denote by $\langle X \rangle$ the smallest Abelian full subcategory of $\mathcal{A}$ which contains X. Equivalently, $\langle X \rangle$ is the full subcategory of $\mathcal{A}$ whose objects are the subquotients of $X^{\oplus n}$ for some n.

Let $\mathcal{A}$ be an arbitrary Abelian k-linear category with a "forgetful" functor $\omega \colon \mathcal{A} \to k\text{-vect}$ (what we are going to call a fiber functor). Following the Freyd-Mitchell theorem[2], one would expect that there exists a k-algebra A such that $\mathcal{A} \cong A\text{-mod}$ over $k\text{-vect}$. The correct way to do this is using coalgebras. We are going to break $\mathcal{A}$ into small bits $\langle X \rangle$ each of which is a category of modules, and glue them together using *co*modules!

Lemma 1. *Let $\mathcal{A}$ be an Abelian k-linear category endowed with a k-linear exact faithful functor $\omega \colon \mathcal{A} \to k\text{-vect}$ and $X \in \mathcal{A}$ be any object. Define the finite-dimensional algebra*

$$A_X := \{\alpha \in \mathrm{End}(\omega(X)) \mid \forall n \geq 1\ \forall Y \subset X^{\oplus n} \text{ subobject}, \alpha^{\oplus n}(Y) \subseteq Y\}.$$

Then $\langle X \rangle \cong A_X\text{-mod}$ naturally over $k\text{-vect}$. Moreover A_X is the endomorphism algebra of the functor $\omega|_{\langle X \rangle}$.

The proof of this lemma is quite technical and is given in [2].

Example 1. In general, if $\mathcal{A} = \langle X \rangle$, then $\mathcal{A} \cong A_X\text{-mod}$. Let G be a finite group and $\mathrm{Rep}_k(G)$ the category of finite dimensional representations of G. Then $\mathrm{Rep}_k(G) \cong \langle kG \rangle$, where kG has the regular representation. The endomorphism algebra is given by the group algebra kG, therefore $\mathrm{Rep}_k(G) \cong kG\text{-mod}$.

If $\langle X' \rangle \subseteq \langle X \rangle$, then the restriction gives a map

$$A_X = \mathrm{End}(\omega|_{\langle X \rangle}) \to \mathrm{End}(\omega|_{\langle X' \rangle}) = A_{X'}$$

which defines an inverse system and we may define $A = \lim A_X$. Since the tensor product does not preserve limits, we are not able to conclude $\mathcal{A} = A\text{-mod}$ (i.e., the limit of A_X-modules is not necessarily a A-modules). On the other hand, since $\otimes$ is itself a colimit, it preserves colimits, and this problem is solved if we work with coalgebras.

1.2 Tannakian Formalism and Coalgebras

Since A_X is finite-dimensional, its dual $B_X := A_X^\vee$ is a coalgebra. In fact, we may define the comultiplication as the natural map

$$\Delta \colon A_X^\vee \xrightarrow{m^\vee} (A_X \otimes A_X)^\vee \xrightarrow{\sim} A_X^\vee \otimes A_X^\vee$$

and the counit as

$$\varepsilon \colon A_X^\vee \xrightarrow{\eta^\vee} k^\vee \xrightarrow{\sim} k.$$

Although less intuitive at first sight, coalgebras are better behaved than algebras. In fact, any coalgebra B is the directed colimit of its finite-dimensional subcoalgebras

[1] Equivalently, when it is enriched over k-vect.

[2] Freyd-Mitchell Theorem: If $\mathcal{A}$ is a small Abelian category, then there exists a ring A and a full, faithful and exact functor $F \colon \mathcal{A} \to A\text{-Mod}$. Moreover, if $\mathcal{A}$ is k-linear, then A is a k-algebra and the functor F is k-linear [1].

B_i (which is clearly false for algebras). This happens because in this case the arrows go the correct way[3], and using this we are able to prove that if $B = \operatorname{colim} B_i$, then the colimit of B_i-modules is a B-module. This leads to the following lemma.

Lemma 2. *Let $\mathcal{A}$ be an Abelian k-linear essentially small category, $\omega\colon \mathcal{A} \to k\text{-}\mathsf{vect}$ a k-linear exact faithful functor and for each $X \in \mathcal{A}$, let $B_X := A_X^\vee$, where A_X is defined as in Lemma 1 and define $B = \operatorname{colim} B_X$. Then there is an equivalence $\mathcal{A} \cong B\text{-}\mathsf{comod}$ over $k\text{-}\mathsf{vect}$.*

This kind of behavior usually appears when working with Hopf algebras, as one may see from the following examples.

Example 2. Let S be a set, and consider $\mathcal{A} = k\text{-}\mathsf{vect}(S)$ the category of finite dimensional S-graded vector spaces. For each finite subset $T \subset S$, we may consider $kT := \bigoplus_{t\in T} k \cdot t$. Then $\langle kT \rangle$ has kT as its "endomorphism" coalgebra, with the operations:

$$\Delta(t) = t \otimes t; \quad \varepsilon(t) = 1 \text{ for all } t \in T.$$

Taking the colimit, we may form kS with the same operations and conclude

$$k\text{-}\mathsf{vect}(S) \cong kS\text{-}\mathsf{comod}.$$

If S is a group, then kS is a Hopf algebra.

Example 3. If G is an affine group scheme over k and $\mathcal{O}_G(G)$ is its coordinate algebra, then $\mathcal{O}_G(G)$ has a natural structure of Hopf algebra (in particular, coalgebra) and $\mathsf{Rep}_k(G) \cong \mathcal{O}_G(G)\text{-}\mathsf{comod}$, so we are not far from our objective. Indeed, we just have to add properties to the category $\mathcal{A} \cong B\text{-}\mathsf{comod}$ to ensure B is a commutative Hopf algebra.

1.3 Tannakian Formalism and Algebraic Groups

The category of representations of a given group has a richer structure than a general Abelian category. For example, such categories inherit a tensor product from vector spaces and an involution given by duality. The generalization of these properties play a major role on the Tannakian formalism.

Definition 1. A symmetric monoidal category consists of an arbitrary category C endowed with the following data:

(i) A bifunctor $\otimes\colon C \times C \to C$;

(ii) A distinguished element $\mathbb{1}$ called the *unit*;

(iii) Natural isomorphisms $\alpha_{X,Y,Z}\colon X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z$ called the *associativity constraints*;

(iv) Natural isomorphisms $\mathbb{1} \otimes X \xrightarrow{\lambda_X} X \xleftrightarrow{\rho_X} X \otimes \mathbb{1}$ caled the *unit constraints*;

(v) Natural isomorphisms $c_{X,Y}\colon X \otimes Y \xrightarrow{\sim} Y \otimes X$ called the *commutativity constraints* (or braiding)

satisfying the following properties

[3] For example, now we are going to have a colimit, which trivially commutes with the tensor product.

- (Triangle) The following diagram commutes

$$\begin{tikzcd}
& (X \otimes \mathbb{1}) \otimes Y \arrow[dl] \arrow[dr] & \\
X \otimes (\mathbb{1} \otimes Y) \arrow[rr] & & X \otimes Y
\end{tikzcd}$$

- (Pentagon) The following diagram with the associativity constraints commutes

$$\begin{tikzcd}
& X \otimes (Y \otimes (Z \otimes W)) \arrow[dl] \arrow[dr] & \\
X \otimes ((Y \otimes Z) \otimes W) \arrow[d] & & (X \otimes Y) \otimes (Z \otimes W) \arrow[d] \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[rr] & & ((X \otimes Y) \otimes Z) \otimes W
\end{tikzcd}$$

- (Hexagon) The following diagram with associativity and commutativity constraints commutes

$$\begin{tikzcd}
& (Y \otimes X) \otimes Z \arrow[dl] & (X \otimes Y) \otimes Z \arrow[l] \arrow[dr] & \\
Y \otimes (X \otimes Z) \arrow[dr] & & & X \otimes (Y \otimes Z) \arrow[dl] \\
& Y \otimes (Z \otimes X) & (Y \otimes Z) \otimes X \arrow[l] &
\end{tikzcd}$$

- (Symmetry) $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}$.

Example 4.

(1) The category of vector spaces is monoidal with the usual tensor product, as is the category of representations of any group (finite or algebraic);

(2) The category of modules over a k-algebra A is monoidal if and only if A is a bialgebra. Analogously, the category of comodules over a k-coalgebra C is monoidal if and only if C is a bialgebra.

(3) Any category with finite (co)products is monoidal with the monoidal stricture given by the (co)product.

Definition 2. Let C and $\mathcal{D}$ be symmetric monoidal categories. A symmetric monoidal functor is a triple (F, ε, μ), where $F\colon C \to \mathcal{D}$ is a functor, $\varepsilon\colon \mathbb{1}_{\mathcal{D}} \xrightarrow{\sim} F(\mathbb{1}_C)$ is an isomorphism and $\mu_{X,Y}\colon F(X) \otimes_{\mathcal{D}} F(Y) \xrightarrow{\sim} F(X \otimes_C Y)$ are natural isomorphisms, and they preserve the unit, associativity and commutativity constraints. If C and $\mathcal{D}$ are k-linear Abelian symmetric monoidal category, we call a functor $\omega\colon C \to \mathcal{D}$ a *fiber functor* if it is a k-linear exact faithful symmetric monoidal functor.

Example 5. Monoidal functors are ubiquitous in Mathematics.

(1) If G is a (finite, or algebraic) group then the forgetful functor $\omega\colon \mathsf{Rep}_k(G) \to k\text{-}\mathsf{vect}$ is symmetric monoidal. The same happens for finite-dimensional bialgebras.

(2) The exterior power $\bigwedge\colon k\text{-}\mathsf{vect} \to k\text{-}\mathsf{Alg}_{\mathrm{gr}}$ is a symmetric monoidal functor (in this example, the the monoidal structure on vector spaces in given by the direct sum). In fact,

$$\bigwedge(V \oplus W) \cong \bigwedge V \otimes \bigwedge W.$$

(3) The Künneth theorem says the total homology $\mathrm{H}_\bullet(-, \mathbb{Q})$ is a monoidal functor.

(4) If X and Y are topological spaces and $f\colon X \to Y$ is a continuous function, then

$$f^*\colon \mathrm{Ab}(Y) \to \mathrm{Ab}(X)$$

is a symmetric monoidal functor.

Definition 3. A symmetric monoidal category C is called *rigid* if for every X in C, there exists an object $X^\vee$ called *dual* of X and morphisms $\mathrm{ev}_X\colon X^\vee \otimes X \to \mathbb{1}$ and $\mathrm{coev}_X\colon \mathbb{1} \to X \otimes X^\vee$ called the *evaluation* and *coevaluation* such that the compositions

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} (X \otimes X^\vee) \otimes X \xrightarrow{\alpha^{-1}_{X,X^\vee,X}} X \otimes (X^\vee \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X$$

$$X^\vee \xrightarrow{\mathrm{id}_{X^\vee} \otimes \mathrm{coev}_X} X^\vee \otimes (X \otimes X^\vee) \xrightarrow{\alpha_{X^\vee,X,X^\vee}} (X^\vee \otimes X) \otimes X^\vee \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^\vee}} X^\vee$$

are the identities.

Example 6. Rigidity is a measure of finiteness, as one may see from the following examples.

(1) The category of vector spaces is **not** rigid, but the category of finite-dimensional vector spaces is rigid. The same happens for representation of groups.

(2) If A is a Hopf algebra over k, then the category A-mod is symmetric rigid. On the other hand, the category of modules of arbitrary dimension is not rigid.

Definition 4. A symmetric monoidal category is called a *Tannakian category* if it is a k-linear Abelian symmetric monoidal rigid category endowed with bilinear tensor product and $\mathrm{End}(\mathbb{1}) = k$.

As the next theorem says, the main property of Tannakian categories is that they are category of representations of group schemes, but before this, let's look at some simple examples.

Example 7.

1. The category of representation of a finite group G is a Tannakian category. In particular, there exists a group scheme G^{alg} called the *algebraic hull* such that $\mathsf{Rep}_k(G) \cong \mathsf{Rep}_k(G^{\text{alg}})$. Since the category is generated by the regular representation, then G^{alg} is a finite group scheme over k. With some computations, one may prove that
$$G^{\text{alg}} = \operatorname{Spec}((kG)^\vee).$$

2. Let X be a connected topological manifold, and $\mathsf{Loc}_k(X)$ the category of local systems on X with coefficients in k. Let $x \in X$ and consider $\omega \colon \mathsf{Loc}_k(X) \to k\text{-}\mathsf{vect}$ be the functor defined by $\mathcal{L} \mapsto \mathcal{L}_x$. This is a Tannakian category and, as is well known,
$$\mathsf{Loc}_k(X) \cong \mathsf{Rep}_k(\pi_1(X, x)).$$

Theorem 1. *Let $\mathcal{A}$ be a Tannakian category and $\omega \colon \mathcal{A} \to k\text{-}\mathsf{vect}$ a fiber functor. Then there exists an affine group scheme G over k such that $\mathcal{A} \cong \mathsf{Rep}_k(G)$ over $k\text{-}\mathsf{vect}$.*

This theorem is proved in [10]. A similar theorem, but with slightly different hypothesis (although equivalent) is prove in [20] and [2].

Remark 1. This affine group is not necessarily algebraic, but it's always *proalgebraic*. Below, we are going to study how to study the geometry of G using the Tannakian category.

1.4 Duality

As is usual in dualities, in our case, it interchanges geometric properties from G with representation-theoretic properties from $\mathsf{Rep}_k(G)$.

Proposition 2. *Let G be an affine group scheme over k.*

1) *The group G is algebraic if and only if there exists $X \in \mathsf{Rep}_k(G)$ generating $\mathsf{Rep}_k(G)$ under direct sums, tensor products, duals, and subquotients.*

2) *Suppose G is algebraic. If G is not connected, then there exists a non-zero $X \in \mathsf{Rep}_k(G)$ such that the subcategory $\langle X \rangle$ is stable under $\otimes$.*

3) *Suppose G is algebraic and connected. If $\operatorname{char} k = 0$ and $\mathsf{Rep}_{\overline{k}}(G_{\overline{k}})$ is semissimple, then G is reductive.*

This proposition is proved in [2] and its proof is not essential for what follows.

Chapter 2

The Affine Grassmannian

The affine Grassmannian appears in the theory of G-bundles. More specifically, it's used when one describes the local properties of G-bundles. We will first give an elementary definition of the affine Grassmanian (particular to the group GL_n) and then we'll give two other viewpoints.

2.1 Lattices

Let A be k-algebra and n be a fixed non-negative integer. The first viewpoint allows us to study lattices in $k(\!(t)\!)$. In what follows, an A-family of lattices of rank n in $k(\!(t)\!)$ is a finitely generated locally free $A[\![t]\!]$-submodule Λ of $A(\!(t)\!)^n$ such that $\Lambda \otimes_{A[\![t]\!]} A(\!(t)\!) = A(\!(t)\!)^n$. If $A = k$, this specializes to the usual definition of a lattice in K^n for $K = k(\!(t)\!)$.

Definition 5. The affine Grassmanian for GL_n, denoted $\mathrm{Gr}_{\mathrm{GL}_n}$, is the k-functor that assigns to each k-algebra A the set of A-families of lattices of rank n in $k(\!(t)\!)$.

This functor is too big to be representable, but it's an ind-projective ind-scheme (in particular, it is ind-proper). The definition and basic properties of ind-schemes are given in the appendix.

Each lattice $\Lambda \subset A(\!(t)\!)^n$ is contained in $t^N A[\![t]\!]^n \subseteq \Lambda \subseteq t^{-N} A[\![t]\!]$ for some $N \in \mathbb{N}$. This family will define a functor $\mathrm{Gr}^{(N)}_{\mathrm{GL}_n}$ which is proper and representable and satisfies $\mathrm{Gr}_{\mathrm{GL}_n} = \bigcup \mathrm{Gr}^{(N)}_{\mathrm{GL}_n}$.

Theorem 2. *The k-functor $\mathrm{Gr}_{\mathrm{GL}_n}$ is an ind-projective ind-scheme over k.*

Proof. We are going to prove the functors

$$\mathrm{Gr}^{(N)}_{GL_n}(A) := \{\Lambda \in \mathrm{Gr}_{\mathrm{GL}_n}(A) \mid t^N A[\![t]\!]^n \subseteq \Lambda \subseteq t^{-N} A[\![t]\!]\}$$

are projective (in particular, proper). This implies that, for $N \leq M$ the inclusions $\mathrm{Gr}^{(N)}_{GL_n} \to \mathrm{Gr}^{(M)}_{\mathrm{GL}_n}$ are proper monomorphisms, or equivalently closed immersions [24, Tag 04XV].

Let $\Lambda_0 := k[\![t]\!]$ and, for each k-algebra A, let $\Lambda_{0,A} := A[\![t]\!]$. For a finite-dimensional k-vector space V, the classical Grassmannian

$$\mathrm{Grass}(V)(A) := \{M \subseteq V \otimes_k A \mid (V \otimes_k A)/M \text{ is a finite locally finite } A\text{-module}\}.$$

is a smooth projective scheme over k [14, Section 8].

Define $V_N := t^{-N}\Lambda_0 / t^N \Lambda_0 \cong k^{2Nn}$, which gives $V_N \otimes A \cong t^{-N}\Lambda_{0,A}/t^N\Lambda_{0,A}$. We claim the map $\mathrm{Gr}^{(N)}_{\mathrm{GL}_n} \to \mathrm{Grass}(V_N)$ defined at A-points by

$$\Lambda \mapsto \Lambda / t^N \Lambda_{0,A}$$

is a closed immersion, which is going to imply the projectiveness of $\mathrm{Gr}^{(N)}_{\mathrm{GL}_n}$.

First of all, let's prove this map is well defined. This means proving that $t^{-N}\Lambda_{0,A}/\Lambda$ is finite locally free. Since Λ is locally free $A[\![t]\!]$-module, then we may suppose it is free after a localization on A. Since $\Lambda[t^{-1}] = A(\!(t)\!)^n$ and $\Lambda[t^{-1}]/\Lambda \cong \bigoplus_{i\geq 1} t^{-i}A^n$, then $A(\!(t)\!)^n/\Lambda$ is a free A-module. It follows that $t^{-N}\Lambda_{0,A}/\Lambda$ is projective or, equivalently, locally free.

Moreover, the image of this morphism are exactly the closed subfunctor of t-invariant submodules. Indeed, let

$$\mathrm{Grass}^t(V_N)(A) := \{M \in \mathrm{Grass}(V_N)(A) \mid t\cdot M \subseteq M\},$$

where t is a k-linear nilpotent operator on V_N. Since the elements of $\mathrm{Gr}^{(N)}_{\mathrm{GL}_n}$ are $A[\![t]\!]$-modules, the image of $\mathrm{Gr}^{(N)}_{\mathrm{GL}_n}$ is clearly t-invariant.

We are going to prove

$$\mathrm{Gr}^{(N)}_{\mathrm{GL}_n} \xrightarrow{\sim} \mathrm{Grass}^t(V_N), \quad \Lambda \mapsto \Lambda/t^N\Lambda_{0,A}$$

which will conclude the proof. This map is clearly injective. For the surjectiveness, let $M \in \mathrm{Grass}^t(V_N)(A)$ and define

$$\Lambda := \ker\left(t^{-N}\Lambda_{0,A} \to t^{-N}\Lambda_{0,A}/t^N\Lambda_{0,A} \cong V_N \otimes A \to (V_N\otimes A)/M\right).$$

By definition, $t^N\Lambda_{0,A} \subseteq \Lambda \subseteq t^{-N}\Lambda_{0,A}$, which implies $\Lambda[t^{-1}] = A(\!(t)\!)^n$. The fact that Λ is a finite locally free $A[\![t]\!]$-module is proved in [26, Lemma 1.1.5] and [22, Theorem 2.2]. □

Many properties of the usual Grassmannian are inherited by the affine Grassmannian, the most important one being probably the Schubert stratification. Most properties of the affine Grassmannian may be proven for an arbitrary group G. There are simple proofs for the $G = \mathrm{GL}_n$ case which are written in [22].

2.2 G-bundles

Let now G be a linear algebraic group over k and also denote by G its pullback to $k(\!(t)\!)$. We may define the Grassmannian for G, denoted by Gr_G, as the functor

$$A \mapsto \left\{(\mathcal{E},\beta) \;\middle|\; \begin{array}{l} \mathcal{E} \text{ is a } G\text{-torsor on } \mathbb{D}_A, \text{ and} \\ \beta : \mathcal{E}|_{\mathbb{D}_A^\times} \cong \mathcal{E}_0|_{\mathbb{D}_A^\times} \text{ is a trivialisation} \end{array}\right\} \Big/ \text{ iso},$$

where $\mathcal{E}_0$ is the trivial torsor.

As for any G-torsor, a trivialisation is the same as a section, so we may define its elements as a G-torsor with a section in $\mathbb{D}_A^\times$.

Observe that $\mathrm{Gr}_G(A)$ is pointed by $[(\mathcal{E}_0,\mathrm{id})]$ and it has a $G(A(\!(t)\!))$-action via $g\cdot[(\mathcal{E},\beta)] := [(\mathcal{E}, g\cdot\beta)]$, Moreover, for $G = \mathrm{GL}_n$, we recover the previous definition.

Indeed, a GL_n-torsor on $\mathbb{D}_A$ corresponds to an n-dimensional vector bundle $\tilde{E} \to \mathbb{D}_A$, which is a locally free $A[\![t]\!]$-module E of rank n, and $\mathcal{E}|_{\mathbb{D}_A^\times}$ corresponds to $E[t^{-1}]$. The trivialisation then means an isomorphism $E[t^{-1}] \cong A(\!(t)\!)^n$, and we may define $\Lambda_{[(\mathcal{E},\beta)]}$ as the image of E under this isomorphism. This shows both definitions of Grassmannian agree.

The ind-representability and ind-properness are not particular to the GL_n.

Theorem 3. *Let G be a reductive linear algebraic group over k. Then the affine Grassmannian $\mathrm{Gr}_G \longrightarrow \mathrm{Spec}(k)$ is ind-proper.*

The proof of this theorem is technical and is given in [22]. We are going to focus on the GL_n case, which we have already proven. This new approach allows us to easily globalize the affine Grassmannian.

Example 8. If G is not reductive, then Gr_G may not be ind-proper, but it is always a separated ind-scheme of ind-finite type and, in particular, a fpqc sheaf. In fact, consider $G = \mathbb{G}_a$. Since $\mathbb{D}_A$ is affine, then [24, Tag 03P6]

$$\mathrm{H}^1_{\text{ét}}(\mathbb{D}_A, \mathbb{G}_a) = \mathrm{H}^1_{\text{Zar}}(\mathbb{D}_A, \mathbb{G}_a) = \mathrm{H}^1(\mathbb{D}_A, \mathcal{O}_{\mathbb{D}_A}) = 0.$$

This means every $\mathbb{G}_a$-torsor is trivial, and a section means simply an element $\mathbb{G}_a(\mathbb{D}_A) = A(\!(t)\!)$. But since we are considering these pairs up to isomorphism,

$$\mathrm{Gr}_{\mathbb{G}_a}(A) = A(\!(t)\!)/A[\![t]\!].$$

This means each class $[(\mathbb{D}_A \times \mathbb{G}_a, \alpha)]$ has a unique representative of the form $\dot{\alpha} = 1 + a_1 t^{-1} + a_2 t^{-2} + \cdots \in A[t^{-1}]$ and we conclude

$$\mathrm{Gr}_{\mathbb{G}_a} \cong \mathbb{A}^\infty_k.$$

2.3 Beauville-Laszlo

The following is a classical result by Beauville and Laszlo [3]. It allows us to "globalize" our definition of Grassmannian.

Theorem 4. *Let X a smooth curve over k, let G be a smooth group scheme over X, let A be a k-algebra and $x \in X_A$ a closed point. Then we have an equivalence of groupoids*

$$\mathrm{Bun}_G(X)(A) \cong \left\{ (\mathcal{E}, \mathcal{F}, \alpha) \;\middle|\; \begin{array}{l} \mathcal{E} \text{ is a } G\text{-torsor over } X_A^\times \\ \mathcal{F} \text{ is a } G\text{-torsor over } \mathbb{D}_{A,x} \\ \beta\colon\ \mathcal{E}|_{\mathbb{D}^\times_{A,x}} \cong \mathcal{F}|_{\mathbb{D}^\times_{A,x}} \text{ is an isomorphism} \end{array} \right\},$$

where $X_A := X \times_k \mathrm{Spec}(A)$ and $\mathbb{D}_{A,x} = \mathrm{Spec}(A \hat{\otimes} \mathcal{O}_x)$.

Under the hypothesis of the theorem, we may define the group $G_x := G \times_X \mathcal{O}_x$ and the functor $\mathrm{Gr}_{G,x}$ on k-Alg as

$$A \mapsto \left\{ (\mathcal{E}, \beta) \;\middle|\; \begin{array}{l} \mathcal{E} \text{ is a } G\text{-torsor on } X_A, \text{ and} \\ \beta : \mathcal{E}|_{X_A^\times} \cong \mathcal{E}_0|_{X_A^\times} \text{ is a trivialisation} \end{array} \right\} \Big/ \text{iso}.$$

As a consequence of the Beauville-Laszlo theorem, we are able to prove that $\mathrm{Gr}_{G,x} \cong \mathrm{Gr}_{G_x}$.

With this moduli definition in mind, we are able to globalize and define the following important functor $\mathrm{Gr}_{G,X}$ called the *Beilinson-Drinfeld Grassmannian*

$$A \mapsto \left\{ (\mathcal{E}, \beta, x) \;\middle|\; \begin{array}{l} x \in X(A) \\ \mathcal{E} \text{ is a } G\text{-torsor on } X_A, \text{ and} \\ \beta : \mathcal{E}|_{X_A \setminus \{x\}} \cong \mathcal{E}_0|_{X_A \setminus \{x\}} \text{ is a trivialisation} \end{array} \right\} \Big/ \text{iso},$$

where $X_A \setminus \{x\}$ is the complement in X_A of the graph $x\colon\ \mathrm{Spec}(A) \longrightarrow X$.

2.4 Loop group

For a given group scheme G over k, we define the *loop group* and *arc group* as the k-functors

$$LG\colon A \mapsto G(A(\!(t)\!))$$

and

$$L^+G\colon A \mapsto G(A[\![t]\!]),$$

the first one being an ind-scheme and the second one a scheme.

Example 9. It's not hard to find the (ind-)scheme representing these functors in particular examples. Indeed, for $G = \mathrm{GL}_n$, observe that we may write an element $M \in \mathrm{GL}_n(A[\![t]\!])$ as a power series $S = S(t) = \sum_{i\geq 0} S_i t^i$, where $S_0 \in \mathrm{GL}_n(A)$ and $S_i \in M_n(A)$ for $i \geq 1$, proving

$$L^+ \mathrm{GL}_n \cong \mathrm{GL}_n \times \left(\prod_{i\geq 1} M_n \right).$$

On the other hand, "having a finite number of nonzero negative coefficients" is not an algebraic condition (i.e. is not given as the zero set of some polynomials) so we are not able to embed $L(\mathrm{GL}_n)$ inside $\prod_{i\in\mathbb{Z}} M_n$ algebraically. The ind-representability of LG follows by a similar argument, but we once again break in smaller pieces. Consider first of all the k-functors

$$M_n^{(N)}(A) := \left\{ S(z) = \sum_{i\geq -N} S_i z^i \mid S_i \in M_n(A) \right\}$$

which is representable by $\prod_{i\geq -N} M_n$. Me may define $\mathrm{GL}_n^{(N)}$ as the subset of $M_n^{(N)} \times M_n^{(N)}$ given by $(S(t), \tilde{S}(t))$ such that $S(t) \cdot \tilde{S}(t) = 1$. Then it follows that

$$L(\mathrm{GL}_n) \cong \bigcup_{N\geq 0} \mathrm{GL}_n^{(N)}$$

is a ind-scheme. A similar proof also works for SL_n.

Example 10. If $G = \mathbb{G}_a$, then $L\mathbb{G}_a(A) = A(\!(t)\!)$ and $L^+\mathbb{G}_a(A) = A[\![t]\!]$. In this case, $L\mathbb{G}_a(A)/L^+\mathbb{G}_a(A) = A[t^{-1}] \cong \mathbb{A}_k^\infty(A)$.

Remark 2. The schemes $\mathbb{D}_k := \mathrm{Spec}(k[\![t]\!])$ and $\mathbb{D}_k^\times := \mathrm{Spec}(k(\!(t)\!))$ should be viewed as a disc and a punctured disc, respectively. Analogously, $\mathbb{D}_A := \mathrm{Spec}(A[\![t]\!])$ and $\mathbb{D}_A^\times := \mathrm{Spec}(A(\!(t)\!))$ should be thought as families of loops/arcs parametrized by R. In this way, the loop group should be viewed as the scheme-theoretically analogue of the loop group in algebraic topology. On the other hand, the arc group should be thought as the contractible loops. It's, then, natural to consider the quotient LG/L^+G. A problem arises because we must be careful with where we are taking this quotient (colimit).

Since we are working with fpqc sheaves, a natural candidate should be the fpqc topology, but it's usually hard to work with it, and we instead work with the étale topology. As we are going to see, this is not going to be a problem, since in this case both quotients coincide.

Surprisingly, we have the following result.

Theorem 5. *Let G be a smooth affine group scheme over k. Then the orbit map $g \mapsto g \cdot e := [(\mathcal{E}, g)]$ induces an isomorphism of étale sheaves*

$$(LG/L^+G)_{\text{et}} \cong \mathrm{Gr}_G\,.$$

In particular, the étale quotient is a fpqc sheaf, and

$$(LG/L^+G)_{\text{et}} \cong (LG/L^+G)_{\text{fpqc}}.$$

Proof. First of all, observe we are considering a map between étale sheaves, therefore to prove surjectiveness we may go to étale cover of k-algebras. Let A be a k-algebra and $[(\mathcal{E}, \alpha)] \in \mathrm{Gr}_G(A)$. We are going to find an étale cover $A \to A'$ such that the $\mathcal{E}|_{\mathbb{D}_{A'}}$ is the trivial G-torsor.

By [24, Tag 02L0, 02VL] the G-bundle $\mathcal{E}$ is affine and smooth over $\mathbb{D}_A$, and the same happens for $\mathcal{E}|_{t=0} \to \mathrm{Spec}(A) \subset \mathbb{D}_A$, where $\mathcal{E}|_{t=0}$ is defined as the restriction of $\mathcal{E} \to \mathbb{D}_A$ along $\{t = 0\} \cong \mathrm{Spec}(A)$. Since $\mathcal{E}|_{t=0}$ is a G-torsor (over the étale topology) A may be covered by finite étale neighborhoods $\mathrm{Spec}(A_i)$ trivialising $\mathcal{E}|_{t=0}$, and the desired affine cover is given by $A' := \prod A_i$.

This means we have a section $\mathrm{Spec}(A') \to \mathcal{E}|_{t=0} \subset \mathcal{E}$. Since $\mathcal{E} \to \mathbb{D}_A$ is smooth, this extends by the infinitesimal lifting criterion to compatible family of sections $\mathrm{Spec}(A'[t]/(t^i)) \to \mathcal{E}$, and since $\mathcal{E}$ is affine, this gives a section $\mathbb{D}_{A'} \to \mathcal{E}$ over $\mathbb{D}_A$. Therefore, we were able to reduce to the case $A = A'$ and $\mathcal{E} = \mathcal{E}_0$ is trivial. In this case, the section defines an element

$$\alpha \in \mathrm{Aut}(\mathcal{E}|_{\mathbb{D}_A^\times}) \cong LG(A),$$

which proves the map is surjective. On the other hand, we must find the stabilizer of $[(\mathcal{E}, e)]$. We have $[(\mathcal{E}, g)] \cong [(\mathcal{E}, e)]$ if and only if g extends to a trivialisation of the whole disc, which means $g \in \mathrm{Aut}(\mathcal{E}) \cong L^+G(R)$, concluding our proof. □

In the complex numbers' case, the rational points have a simple description.

Corollary 1. *Let G be a smooth affine group scheme over a separably closed field k. Then the previous isomorphism gives*

$$\mathrm{Gr}_G(k) \cong G(k(\!(t)\!))/G(k[\![t]\!]).$$

Proof. Since k is separably closed, the Galois cohomology $H^1_{\text{ét}}(\mathrm{Spec}(k), L^+G)$ is trivial, and the result follows. □

In particular, we have $\mathrm{Gr}_{GL_1}(\mathbb{C}) = \mathbb{Z}$.

2.5 Decompositions

Let G be a connected reductive group over $\mathbb{C}$ and fix a triple $G \supset B \supset T$, where B is a Borel subgroup and T is a maximal torus. We will denote by N the unipotent radical of B. We denote by $\Phi := \Phi(G, T)$ the root system of (G, T), by $\Phi_+ := \Phi_+(G, B, T)$ the positive roots and by $\Phi_s := \Phi_s(G, B, T)$ the simple roots. For a root α, we denote $\alpha^\vee$ the corresponding coroot.

A given cocharacter $\nu \in X_\bullet(T)$, defines a morphism $\mathbb{C}(\!(t)\!)^\times \to LT$. The image of t is denoted t^ν, and we let $L_\nu := t^\nu L^+G$ be the corresponding point in the affine Grassmannian. There are canonical points which allow us to decompose the affine Grassmannian into smaller pieces called Schubert cells.

Proposition 3 (Cartan Decomposition)**.** *There is a stratification $\mathcal{S}$ of Gr_G given by*

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in X_\bullet(T)^+} \mathrm{Gr}_G^\lambda, \quad \textit{where} \quad \mathrm{Gr}_G^\lambda := L^+G \cdot L_\lambda.$$

where each Schubert cell Gr_G^λ is an affine bundle over the partial flag variety G/P_λ where P_λ is the parabolic subgroup of G containing B and corresponding to the subset of simple roots $\{\alpha \in \Phi_s \mid \langle \lambda, \alpha \rangle = 0\}$. Moreover, their dimension is given by

$$\dim\left(\mathrm{Gr}_G^\lambda\right) = \langle 2\rho, \lambda \rangle$$

and their closure is well-behaved

$$\overline{\mathrm{Gr}_G^\lambda} = \mathrm{Gr}_G^{\leq\lambda} := \bigsqcup_{\substack{\eta \in X_\bullet(T)^+ \\ \eta \leq \lambda}} \mathrm{Gr}_G^\eta.$$

The Schubert cells Gr_G^λ are always quasi-projective smooth subvarieties. On the other hand, the Schubert varieties $\mathrm{Gr}_G^{\leq\lambda}$ are projective but not necessarily smooth (in particular, they are compact).

Remark 3. The connected component of Gr_G are given by

$$\mathrm{Gr}_G^c := \bigsqcup_{\substack{\lambda \in X_\bullet(T) \\ \lambda + \Phi^\vee = c}} \mathrm{Gr}_G^\lambda$$

for each $c \in X_\bullet(T)/\Phi^\vee$. Since $\langle \rho, \lambda \rangle \in \mathbb{Z}$ for each $\lambda \in \Phi^\vee$, it follows the parity of the dimensions of the Schubert varieties is constant in each connected component.

Example 11. This decomposition was first observed by Lusztig on [17] studying characters of $\mathrm{GL}_n(\mathbb{F}_q)$. In our case, let $G = \mathrm{GL}_n$ and consider the description of $\mathrm{Gr}_{\mathrm{GL}_n}$ as the lattices. If $\lambda = (r, 0, \ldots, 0)$, then

$$\mathrm{Gr}_G^{\leq\lambda}(A) = \{\Lambda \subset \Lambda_{0,A} \mid \mathrm{rk}_A(\Lambda_{0,A}/\Lambda) = r\}.$$

If $r = n$, and $\mathcal{N}_n$ denote the set of $n \times n$ nilpotent matrices, then we have an embedding $\mathcal{N}_n \longrightarrow \mathrm{Gr}_G^{\leq\lambda}$ given by

$$M \mapsto (tI_n - M)\Lambda_0.$$

Since M is nilpotent, then $\det(tI_n - M) = t^n$, and the function is well-defined. By [26, Lemma 2.3.12], this is an open embedding, and, in particular, it gives a compactification of $\mathcal{N}_n$. Observe that this function simply means we are taking Λ such that multiplication by t acts as M on $\Lambda_0/\Lambda \cong \mathbb{C}^n$.

Moreover, the dominant cocharacters $\mu \leq \lambda$ are exactly the partitions of n, which also indexes the collection of equivalence classes $\mathcal{O}_\mu$ of nilpotent matrices (by Jordan canonical form). Moreover, $\mathcal{O}_\mu$ is sent into Gr_G^μ under this embedding. In fact, observe that $L_\mu = (t^{\mu_1}\mathbb{C}[\![t]\!], t^{\mu_2}\mathbb{C}[\![t]\!], \ldots, t^{\mu_n}\mathbb{C}[\![t]\!])$. In particular, multiplication by t acts as

$$\begin{pmatrix} J_{\mu_1} & & & \\ & J_{\mu_2} & & \\ & & \ddots & \\ & & & J_{\mu_n} \end{pmatrix}$$

on the basis $(e_1, t \cdot e_1, \dots t^{\mu_1 - 1} e_1, e_2, t \cdot e_2, \dots)$ of Λ_0 / L_μ. Moreover, this action is invariant if we change L_μ by some other $S \cdot L_\mu$ for $S \in \mathrm{GL}_n(\mathbb{C}[\![t]\!])$. This means we cannot have an element $M \in \mathcal{O}_\mu$ going to Gr_G^ν for $\mu \neq \nu$, and therefore $\mathcal{O}_\lambda \subset \mathrm{Gr}_G^\lambda$.

Another important decomposition from the theory of algebraic groups is given by the Iwasawa decomposition, whose strata are also called semi-infinite orbits.

Proposition 4 (Iwasawa Decomposition)**.** *There is a decomposition*

$$\mathrm{Gr}_G = \bigsqcup_{\mu \in X_\bullet(T)} S_\mu, \quad \textit{where} \quad S_\mu := LN \cdot L_\mu.$$

Using Kac-Moody algebras, we may embed Gr_G in an (infinite dimensional) projective space $\mathbb{P}(V)$, which allows us to control its geometry. The following is an important result proved in [2].

Proposition 5. *For any cocharacter $\mu \in X_\bullet(T)$, the closure is well-behaved*

$$\overline{S_\mu} = \bigsqcup_{\nu \leq \mu} S_\nu.$$

Moreover, its boundary is given by the intersection with a hyperplane. More precisely, there is a (infinite-dimensional) vector space V, a closed embedding $\Psi \colon \mathrm{Gr}_G^{\mu + \Phi^\vee} \to \mathbb{P}(V)$ and a hyperplane H_μ of $\mathbb{P}(V)$ such that

$$\partial S_\mu = \overline{S_\mu} \cap \psi^{-1}(H_\mu).$$

Everything can be done for the Borel subgroup B^- opposite to B and its nilpotent radical N^-. We may define $T_\mu := LN^- \cdot L_\mu$ and it has the same closure properties. Finally, this last lemma is going to be useful for a induction argument later.

Lemma 3. *Let μ, $\nu \in X_\bullet(T)$. Then $\overline{S_\mu} \cap \overline{T_\nu} = \varnothing$ except $\nu \leq \mu$, and $\overline{S_\mu} \cap \overline{T_\mu} = \{L_\mu\}$.*

Chapter 3

Perverse Sheaves

Homological Algebra is usually done in the category $D(\mathcal{A})$ for some Abelian category $\mathcal{A}$. After giving the definitions of triangulated category and t-structure, we study some properties of perverse sheaves, which is important in the study of non-smooth schemes.

3.1 Triangulated Categories

Triangulated categories were defined by Verdier in his thesis, to formalize a generalization of Serre's duality nowadays called Grothendieck duality. Similar ideas were already being proposed and probably Verdier's most important contribution was the octahedral axiom. This obscure axiom is the responsible for the existence of the heart of a t-structure.

Since its creation, triangulated categories are a very important tool to describe problems in Algebraic Geometry, Representation Theory and many other areas: Grothendieck Duality, Bondal-Orlov Theorem, Broué Conjecture, to name a few.

Definition 6. A *triangulated category* consists of the following data:

1) An additive category $\mathcal{D}$;

2) An automorphism $[1] \colon \mathcal{D} \to \mathcal{D}$;

3) A family of distinguished triangles

$$X \to Y \to Z \to X[1]$$

satisfying the following properties:

TR1) For any X and any map $f \colon X \to Y$:

- A triangle isomorphic to a distinguished triangle is also distinguished;
- The triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$ is always distinguished;
- Every morphism $X \xrightarrow{f} Y$ embeds in a triangle

$$X \xrightarrow{f} Y \to Z \to X[1];$$

TR2) A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is distinguished if and only if the triangle $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y$ is also distinguished;

TR3) Given a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] & X[1] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

in $\mathcal{D}$ whose rows are distinguished triangles, there exists a morphism $h\colon Z \to Z'$ such that the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

commutes.

TR4) (Octahedral axiom) Suppose we are given the following three distinguished triangles:

$$X \xrightarrow{f} Y \longrightarrow Z' \longrightarrow X[1],$$

$$Y \xrightarrow{g} Z \longrightarrow X' \longrightarrow Y[1],$$

$$X \xrightarrow{g\circ f} Z \longrightarrow Y' \longrightarrow X[1].$$

Then there exists a distinguished triangle

$$Z' \longrightarrow Y' \longrightarrow X' \longrightarrow Z'[1]$$

making the diagram

$$\begin{tikzcd}
X \arrow[rr, bend left, "g\circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, bend left] \arrow[dr] & & X' \arrow[rr, dashed, bend left] \arrow[dr] & & Z'[1] \\
& Y \arrow[ur, "g"'] \arrow[dr] & & Y' \arrow[ur, dashed] \arrow[dr] & & Y[1] \arrow[ur] & \\
& & Z' \arrow[ur, dashed] \arrow[rr, bend right] & & X[1] \arrow[ur] & &
\end{tikzcd}$$

commute.

As usual, these axioms are not independents. For example, May have proved [19, Lemma 2.2] that TR3) is a consequence of the other axioms.

Some easy consequences follows from these axioms. For example, the composite $g \circ f$ of two arrows from a distinguished triangle is 0.

Example 12.

(1) For a scheme X, the category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X is an Abelian category with enough injectives. We are able to define the derived

category $D(\mathrm{QCoh}(X))$ and the injective resolutions allow us to compute the derived functors. Similarly, we the category of $\mathcal{O}_X$-modules is also Abelian and has enough injectives, and we may define the derived category $D(\mathcal{O}_X\text{-}\mathsf{Mod})$. We may similarly define $D_{\mathsf{qc}}(\mathcal{O}_X\text{-}\mathsf{Mod})$ as the subcategory of chain complexes with quasi-coherent cohomology. When X is quasi-compact and separated, then

$$D(\mathrm{QCoh}(X)) \cong D_{\mathsf{qc}}(\mathcal{O}_X\text{-}\mathsf{Mod}).$$

(2) For a finite group G, the *stable category* $kG\text{-}\underline{\mathsf{mod}} := kG\text{-}\mathsf{mod}/kG\text{-}\mathsf{proj}$ admits a canonical triangulated structure, whose translation functor is given by the inverse of the Heller operator Ω, defined by an exact sequence

$$0 \to \Omega V \to P \to V \to 0,$$

where P is a projective module [15].

Our main example of triangulated category is going to be the category of constructible chain complexes, which are built of local systems. They were first defined in Algebraic Topology to work with twisted coefficients (i.e. endowed with an action of the fundamental group of the space). If X is a well-behaved space, then

$$\mathsf{Loc}_k(X) \cong \mathsf{Rep}_k(\pi_1(X, x)).$$

Definition 7. Let X be a topological space. A local system on X is a locally constant sheaf with finite dimensional stalks.

Definition 8. Let X be a topological space. A *stratification* of X is a collection of locally closed subsets X_i such that $X = \bigsqcup_i X_i$ and such that $\overline{X_i}$ is the unions of some X_j.

Intuitively, each X_i should be a smooth manifold and we would be able to build X gluing smooth manifolds.

Definition 9. Let X be a topological space and $\mathcal{S}$ a stratification of X.

- We call a sheaf $\mathcal{F}$ *constructible* with respect to $\mathcal{S}$ if for each $S \in \mathcal{S}$, $\mathcal{F}|_S$ is a local system on X. We call a sheaf $\mathcal{F}$ constructible if it is contructible with respect to some stratification $\mathcal{S}$.
- A complex of sheaves $\mathcal{F}$ is constructible with respect to $\mathcal{S}$ if $\mathrm{H}^i(\mathcal{F})$ is constructible with respect to $\mathcal{S}$ for every $i \in \mathbb{Z}$. A complex of sheaves $\mathcal{F}$ is constructible if is is constructible with respect to some stratification$\mathcal{S}$.

The full subcategory of $D^b(X, k)$ of constructible complexes is denoted by $D^b_c(X, k)$ and the full subcategory of $\mathcal{S}$-constructible complexes is denotes by $D^b_{\mathcal{S}}(X, k)$.

Let X be a complex algebraic variety and consider X^{an} be the set $X(\mathbb{C})$ endowed with the analytic topology. A non-trivial result [16, Theorem 4.5.8] is the fact that $D^b_c(X^{\mathrm{an}}, k)$ is stable under the usual operations of (complexes of) sheaves: f_*, f^*, $f_!$, $f^!$, $\otimes^{\mathbf{L}}$ and $\mathbf{R}\underline{\mathrm{Hom}}$. In particular, it's also stable under $\boxtimes$.

Therefore the dualizing complex $\omega_{X^{\mathrm{an}}} := f^!_{X^{\mathrm{an}}}\underline{k}$ (where $f_{X^{\mathrm{an}}} \colon X^{\mathrm{an}} \to \{*\}$) is a constructible sheaf, and we are able to define the *Verdier duality functor*

$$\mathbf{D}_{X^{\mathrm{an}}} := \mathbf{R}\underline{\mathrm{Hom}}(-, \omega_{X^{\mathrm{an}}}).$$

The following proposition is also a part of [16, Theorem 4.5.8].

Proposition 6. *If X is an algebraic variety, then $\mathbf{D}_{X^{\mathrm{an}}} \circ \mathbf{D}_{X^{\mathrm{an}}} = \mathrm{Id}$. Moreover, if Y is another complex algebraic variety and $f = \varphi^{\mathrm{an}} \colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ for some morphism φ of algebraic varieties, then*

$$f_! = \mathbf{D}_{Y^{\mathrm{an}}} \circ f_* \circ \mathbf{D}_{X^{\mathrm{an}}}$$
$$f^! = \mathbf{D}_{X^{\mathrm{an}}} \circ f^* \circ \mathbf{D}_{Y^{\mathrm{an}}}.$$

3.2 t-structures

The t-structure were introduced together with perverse sheaves to generalize results from smooth manifolds and varieties to pseudo-manifolds and algebraic varieties. For example, they allow us to generalize Poincaré duality and the decomposition theorem for non-smooth manifolds.

They are also used in Representation Theory where an equivalence between the derived categories of two algebras A and B preserving the standard t-structure ensures A and B are Morita equivalent. They are a powerful tool to bring derived information to the Abelian world.

Definition 10. Let $\mathcal{D}$ be a triangulated category. A t-structure is a pair of a full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ such that:

- If we denote $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n]$, then $\mathcal{D}^{\leq -1} \subseteq \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq -1} \supseteq \mathcal{D}^{\geq 0}$;
- $\mathrm{Hom}(\mathcal{D}^{\leq -1}, \mathcal{D}^{\geq 0}) = 0$;
- For each $X \in \mathcal{D}$, there exists $X' \in \mathcal{D}^{\leq -1}$ and $X'' \in \mathcal{D}^{\geq 0}$ and a distinguished triangle
$$X' \to X \to X'' \to X'[1].$$

Example 13. Let $\mathcal{A}$ be an abelian category and let $\mathcal{D} = D(\mathcal{A})$. The standard t-structure is given by

$$\mathcal{D}^{\leq 0} := \{A \in \mathcal{D} \mid \mathrm{H}^p(A) = 0 \text{ for } p > 0\}$$
$$\mathcal{D}^{\geq 0} := \{A \in \mathcal{D} \mid \mathrm{H}^p(A) = 0 \text{ for } p < 0\}.$$

Observe that $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} \cong \mathcal{A}$ is an Abelian category. Moreover, if $\tau_{\leq -1}$ and $\tau_{\geq 0}$ are the usual truncations, then $\tau_{\leq -1}A \in \mathcal{D}^{\leq -1}$ and $\tau_{\geq 0}A \in \mathcal{D}^{\geq 0}$. Moreover, as is well known
$$\tau_{\leq -1}A \to A \to \tau_{\geq 0} \to \tau_{\leq -1}A[1].$$

Definition 11. Let $\mathcal{D}$ be a triangulated category with t-structure given by $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Its *heart* is given by $\mathcal{D}^{\heartsuit} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$.

An important and non-trivial result is the following theorem [16, Theorem 8.1.9].

Theorem 6. *The heart of a t-structure is an Abelian category. Moreover, if*

$$0 \to X' \to X \to X'' \to 0$$

is exact in $\mathcal{D}^{\heartsuit}$, then we have a distinguished triangle in $\mathcal{D}$

$$X' \to X \to X'' \to X'[1].$$

Let $\mathcal{D}$ be a triangulated category with t-structure given by $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Moreover, let $n \in \mathbb{Z}$ and let $i\colon \mathcal{D}^{\leq n} \to \mathcal{D}$ be the inclusion. Then i has a right adjoint $\tau^{\leq n}$, which may be seen by the Adjoint Functor Theorem [24, Tag 0A8G]. Analogously, the inclusion $i'\colon \mathcal{D}^{\geq n} \to \mathcal{D}^{\geq 0}$ has a *left* adjoint called $\tau^{\geq n}$.

If $X \in \mathcal{D}$ is any element, then it's not hard to prove [16, Proposition 8.1.5]

$$\tau^{\leq n}X \to X \to \tau^{\geq n+1}X \to \tau^{\leq n}X[1].$$

Moreover, the elements X' and X'' given by the third property of a t-structure are unique up to isomorphism, which means $X' \cong \tau^{\leq -1}X$ and $X'' \cong \tau^{\geq 0}X$.

Proposition 7. *Let $\mathcal{D}$ be a triangulated category endowed with a t-structure whose heart is given by $\mathcal{D}^{\heartsuit}$. The functor $\mathrm{H}^0 := \tau^{\leq 0}\tau^{\geq 0}\colon \mathcal{D} \to \mathcal{D}^{\heartsuit}$ is cohomological. For $n \in \mathbb{Z}$, we also define $\mathrm{H}^n(X) := H^0(X[n]) = \tau^{\leq n}\tau^{\geq n}X$.*

Finally, the last definition is used to define the intermediate extension and describes our philosophy that t-structure allows us to go from the derived world to the Abelian world.

Definition 12. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two triangulated categories endowed with t-structures whose hearts are $\mathcal{D}_1^{\heartsuit}$ and $\mathcal{D}_2^{\heartsuit}$, respectively. Let $\varepsilon_1\colon \mathcal{D}_1^{\heartsuit} \to \mathcal{D}_1$ be the inclusion and $F\colon \mathcal{D}_1 \to \mathcal{D}_2$ be a triangulated functor. Then we define the additive functor

$${}^pF := \mathrm{H}^0 \circ F \circ \varepsilon_1\colon \mathcal{D}_1^{\heartsuit} \to \mathcal{D}_2^{\heartsuit}.$$

3.3 Perversity

The perverse t-structure was defined to study the intersection cohomology. Moreover, the perverse sheaves should be stable under the duality functor. This motivates the following definition.

Definition 13. Let X be a complex algebraic variety. We define a t-structure on $D_c^b(X^{\mathrm{an}}, k)$ by

$${}^pD_c^{\leq 0} := \{\mathscr{F} \in D_c^b(X^{\mathrm{an}}, k) \mid \operatorname{supp} \mathscr{H}^{-i}(\mathscr{F}) \leq i \text{ for every } i \in \mathbb{Z}\}$$

and ${}^pD_c^{\geq 0} := \mathbf{D}_X({}^pD_c^{\leq 0})$, which is callted *perverse t-structure*.

The heart of this t-structure is called the category of *perverse sheaves* and denoted by $\mathrm{P}(X, k)$.

Remark 4. Perverse sheaves are not sheaves. However, the correspondence $X \mapsto \mathrm{P}(X, k)$ defines a stack on the Zariski topology. Therefore, perverse sheaves may still be glued, although in a more complicated way.

If $\mathcal{S}$ a stratification of X, this also defines a t-structure $({}^pD_{\mathcal{S}}^{\leq 0}, {}^pD_{\mathcal{S}}^{\geq 0})$ on $D_{\mathcal{S}}^b(X^{\mathrm{an}}, k)$, whose heart is denoted by $\mathrm{P}_{\mathcal{S}}(X, k)$. We are going to work with this one, since it is more closely related to the equivariant perverse sheaves.

Proposition 8. *Let X be a complex algebraic variety stratified by $\mathcal{S}$.*

1. *A complex $\mathscr{F} \in D^b(X, k)$ belongs to ${}^pD_{\mathcal{S}}^{\leq 0}$ if and only if for all $S \in \mathcal{S}$*

$$(i_S)^*\mathscr{F} \in D^{\leq -\dim S}(S, k);$$

2. *A complex $\mathscr{F} \in D^b(X,k)$ belongs to ${}^pD^{\geq 0}_{\mathcal{S}}$ if and only if for all $S \in \mathcal{S}$*

$$(i_S)^!\mathscr{F} \in D^{\geq -\dim S}(S,k).$$

Let X be a complex algebraic variety, S be a subvariety of X, and let $i\colon S \hookrightarrow X$ be the inclusion. For a complex of sheaves $\mathscr{F} \in D^b_c(S,k)$, we usually have two notions of extensions $i_*\mathscr{F}$ and $i_!\mathscr{F}$. When working with perverse sheaves, we must use the perverse versions ${}^pi_*$ and ${}^pi_!$ defined at Definition 12. While the morphism $i_! \hookrightarrow i_*$ is a monomorphism, this is not true[1] for the morphism ${}^pi_! \to {}^pi_*$. We denote by $i_{!*}$ the image of ${}^pi_!$ in ${}^pi_*$ and call it the *middle extension*.

A classical result of perverse sheaves says $\mathrm{P}_{\mathcal{S}}(X,k)$ is both Noetherian and Artinian, meaning it has finite length. Its simple objects may be indexed by pairs $(S,\mathscr{L})$ where $S \in \mathcal{S}$ and $\mathscr{L}$ is a local system on S and are given by $(i_S)_{!*}\mathscr{L}[\dim S]$. More concretely, $(i_S)_{!*}\mathscr{L}[\dim S]$ is the unique element of $\mathscr{F} \in \mathrm{P}_{\mathcal{S}}(X,k)$ such that, if $i\colon \overline{S}\backslash S \hookrightarrow X$ is the inclustion, then

$$\operatorname{supp}\mathscr{F} \subseteq \overline{S},\ \mathscr{F}|_S = \mathscr{L}[-\dim S],\ i^*\mathscr{F} \in {}^pD^{\leq -1}(\overline{S}\backslash S,k),\ i^!\mathscr{F} \in {}^pD^{\geq 1}(\overline{S}\backslash S,k).$$

Such an element is denoted by $\mathbf{IC}(S,\mathscr{L})$ and is called intersection cohomology sheaf on $\overline{S}$ with coefficients in $\mathscr{L}$.

The next definition is an important property of maps which is important in Geometric Representation Theory. The proposition following its definition shows its importance for us.

Definition 14. Let $(X,\mathcal{S})$ and $(Y,\mathcal{T})$ be two stratified complex algebraic varieties. A proper map $f\colon X \to Y$ is called stratified semi-small if for every $S \in \mathcal{S}$, the image $f(S)$ is a union of stratas $T \in \mathcal{T}$ and if for every $S \in \mathcal{S}$ and $T \subset f(S)$ and for any $y \in T$, we have

$$\dim(S \cap f^{-1}(y)) \leq \frac{1}{2}(\dim S - \dim T).$$

Moreover, we say f is locally trivial if for every $S \in \mathcal{S}$ and for every $T \in \mathcal{T}$ we have the map $f\colon S \cap f^{-1}(T) \to T$ is a Zariski locally trivial fibration.

Remark 5. The usual notion of a map $f\colon X \to Y$ being semi-small is given by the inequality

$$\dim(X \times_Y X) \leq \dim X.$$

Our definition is a refinement of this one for the stratified world.

Proposition 9. *Let $(X,\mathcal{S})$ and $(Y,\mathcal{T})$ be two stratifies complex algebraic varieties and let $f\colon X \to Y$ be a stratified semi-small and locally trivial map. If $\mathscr{A} \in \mathrm{P}_{\mathcal{S}}(X,k)$, then $f_*(\mathscr{A}) \in \mathrm{P}_{\mathcal{T}}(Y,k)$.*

For completeness, we are also going to define the equivariant perverse sheaves and give its basic properties. It is possible to begin defining equivariant sheaves, then the equivariant derived category $D_G(X,k)$ to finally define the equivariant perverse sheaves, this is done in [6]. We are going to follow a more direct approach, whose equivalence is proved in [2].

[1] An interesting example is found when working the cone $X = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = z^2\}$ stratified by $Z = \{(0,0,0\}$ and $U = X\backslash Z$. In this case, $j_!\underline{k}_U[1] \in \mathrm{P}_{\mathcal{S}}(X,k)$ and $j_*\underline{k}_U[1] = \underline{k}_X[1] \in \mathrm{P}_{\mathcal{S}}(X,k)$, and the morphism $j_!\underline{k}_U[1] \to \underline{k}_X[1]$ is surjective, but not injective [7, Example 10.4].

Let G be a complex algebraic group and let X be the set of complex points of an algebraic variety endowed with a stratification $\mathcal{S}$ of G-invariant strata. Moreover, let

$$a, p\colon G \times X \to X \quad e\colon X \to G \times X$$

be defined by

$$a(g, x) = gx, \quad p(g, x) = x \quad \text{and} \quad e(x) = (1, x).$$

An equivariant perverse sheaf is defined as a pair $(\mathcal{F}, \varphi)$ where $\mathcal{F} \in \mathrm{P}_{\mathcal{S}}(X, k)$ and $\varphi\colon a^*\mathcal{F} \cong p^*\mathcal{F}$ is an isomorphism such that

$$e^*(\varphi) = \mathrm{id}, \quad \text{and} \quad (m \times \mathrm{id}_X)^*(\varphi) = (p_{23})^*(\varphi) \circ (\mathrm{id}_H \times a)^*(\varphi),$$

where $m\colon G \times G \to G$ is the multiplication and $p_{23}\colon G \times G \times X \to G \times X$ is the projection on the last two coordinates. A morphism of equivariant perverse sheaves is a morphism of the corresponding sheaves making a natural square commutative. The category of equivariant perverse sheaves is denoted by $\mathrm{P}_G(X, k)$.

As a final comment, we recall the following result from [6, 2.6.2], whose analogue for constructible sheaves is going to be used in the next chapter.

Proposition 10. *Let G be a topological group, $H \triangleleft G$ be a closed normal subgroup and let $K = G/H$ be the quotient group. Let X be a topological space in which G acts and such that the H-action is free. Then*

$$D_K(X/H, k) \cong D_G(X, k).$$

Chapter 4

The Satake Category

The Satake category is defined as the category of L^+G equivariant sheaves in the Grassmannian $\mathrm{Sat}_G := \mathrm{P}_{L^+G}(\mathrm{Gr}_G, k)$. As we are going to prove shortly, if $\mathcal{S}$ is the stratification given by the Schubert cells then $\mathrm{P}_{L^+G}(\mathrm{Gr}_G, k) \cong \mathrm{P}_{\mathcal{S}}(\mathrm{Gr}_G, k)$, and so we are going to interchange them when needed. Since Gr_G is infinite-dimensional, we do not define the perverse sheaves in the usual way. Instead, we define

$$\mathrm{P}_{L^+G}(\mathrm{Gr}_G, k) := \operatorname{colim}_\lambda \mathrm{P}_{L^+G}(\mathrm{Gr}_G^{\leq\lambda}, k),$$

and then the finitess properties of $\mathrm{P}_{L^+G}(\mathrm{Gr}_G^{\leq\lambda}, k)$ are inherited by $\mathrm{P}_{L^+G}(\mathrm{Gr}_G, k)$. We define similarly the Abelian category $\mathrm{P}_{\mathcal{S}}(\mathrm{Gr}_G, k)$.

4.1 Semissimplicity of the Satake Category

Our variety is going to be $X = \mathrm{Gr}_G$ and its stratification $\mathcal{S}$ is the Schubert stratification. Since the Schubert cells Gr_G^λ are affine bundles over partial flags varieties, one may use the long exact sequence of homotopy groups associated with a fibration, to show they are simply connected. Therefore, there is a unique simple local system $\mathcal{L} = \underline{k}$. We are going to denote $\mathbf{IC}_\lambda$ the corresponding intersection cohomology sheaf.

Using what we said before, we see it's enough to prove the following.

Theorem 7. *For any $\lambda, \mu \in X^\bullet(T)^+$, we have*

$$\mathrm{Hom}_{D^b_{\mathcal{S}}(\mathrm{Gr}_G,k)}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) = 0.$$

The proof may be divided into three parts, the last one being more technical.

Proof.
First case: $\lambda = \mu$**.** Consider the following diagram

$$\begin{tikzcd} \mathrm{Gr}_G^\lambda \arrow[r, "j"] \arrow[dr, "j_\lambda"'] & \overline{\mathrm{Gr}_G^\lambda} \arrow[d, "i_\lambda"] & \overline{\mathrm{Gr}_G^\lambda} \setminus \mathrm{Gr}_G^\lambda \arrow[l, "i"'] \\ & \mathrm{Gr}_G & \end{tikzcd}$$

in which every map is a embedding and let $\mathcal{F} := (i_\lambda i)^* \mathbf{IC}_\lambda$. By the third property of $\mathbf{IC}_\lambda$, $\mathcal{F}$ is concentrated in negative perverse degrees, while $(i_\lambda i)^! \mathbf{IC}_\lambda$ is concentrated in positive perverse degrees. Therefore it follows that

$$\mathrm{Hom}_{D^b_{\mathcal{S}}(\overline{\mathrm{Gr}_G^\lambda} \setminus \mathrm{Gr}_G^\lambda, k)}(\mathcal{F}, (i_\lambda i)^! \mathbf{IC}_\lambda[1]) = 0.$$

Since the functor $\mathrm{Hom}_{D^b_{\mathcal{S}}(\mathrm{Gr}_G,k)}((i_\lambda)_!-, \mathbf{IC}_\lambda[1])$ is cohomological, applying to the distinguished triangle

$$j_!j^*\left(\mathbf{IC}_\lambda|_{\overline{\mathrm{Gr}^\lambda_G}}\right) \to \left(\mathbf{IC}_\lambda|_{\overline{\mathrm{Gr}^\lambda_G}}\right) \to i_!i^*\left(\mathbf{IC}_\lambda|_{\overline{\mathrm{Gr}^\lambda_G}}\right) \xrightarrow{[1]} \tag{4.1}$$

gives the exact sequence

$$\begin{aligned}\mathrm{Hom}_{D^b_{\mathcal{S}}(\mathrm{Gr}_G,k)}((i_\lambda i)_!\mathcal{F}, \mathbf{IC}_\lambda[1]) \to \mathrm{Hom}_{D^b_{\mathcal{S}}(\mathrm{Gr}_G,k)}(\mathbf{IC}_\lambda, \mathbf{IC}_\lambda[1])&\\ \to \mathrm{Hom}_{D^b_{\mathcal{S}}(\mathrm{Gr}_G,k)}((j_\lambda)_!\underline{k}_{\mathrm{Gr}^\lambda_G}[\dim \mathrm{Gr}^\lambda_G], \mathbf{IC}_\lambda[1]),&\end{aligned}$$

where we have used the second property of simple objects above. As we have proved in (4.1), the first object is zero. Moreover, the last object is given by

$$\begin{aligned}&\mathrm{Hom}_{D^b_{\mathcal{S}}(\mathrm{Gr}_G,k)}((j_\lambda)_!\underline{k}_{\mathrm{Gr}^\lambda_G}[\dim \mathrm{Gr}^\lambda_G], \mathbf{IC}_\lambda[1])\\ &= \mathrm{Hom}_{D^b_{\mathcal{S}}(\mathrm{Gr}_G,k)}(\underline{k}_{\mathrm{Gr}^\lambda_G}[\dim \mathrm{Gr}^\lambda_G], (j_\lambda)^!\mathbf{IC}_\lambda[1])\\ &= \mathrm{Hom}_{D^b_{\mathcal{S}}(\mathrm{Gr}_G,k)}(\underline{k}_{\mathrm{Gr}^\lambda_G}, \underline{k}_{\mathrm{Gr}^\lambda_G}[1])\\ &= \mathrm{H}^1(\mathrm{Gr}^\lambda_G, k),\end{aligned}$$

which is zero, since Gr^λ_G is an affine bundle over a complex partial flag variety, and therefore has only cohomology in even degrees. Hence we conclude

$$\mathrm{Hom}_{D^b_{\mathcal{S}}(\mathrm{Gr}_G,k)}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) = 0.$$

Second case: Neither $\mathrm{Gr}^\lambda_G \subset \overline{\mathrm{Gr}^\mu_G}$ nor $\mathrm{Gr}^\mu_G \subset \overline{\mathrm{Gr}^\lambda_G}$.

Let $i_\mu \colon \overline{\mathrm{Gr}^\mu_G} \to \mathrm{Gr}_G$ be the inclusion. Since $\mathbf{IC}_\mu$ is supported in $\overline{\mathrm{Gr}^\mu_G}$, it satisfies $\mathbf{IC}_\mu = (i_\mu)_*(i_\mu)^*\mathbf{IC}_\mu$, and therefore

$$\mathrm{Hom}_{D^b_{\mathcal{S}}(\mathrm{Gr}_G,k)}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) = \mathrm{Hom}_{D^b_{\mathcal{S}}(\overline{\mathrm{Gr}^\mu_G},k)}((i_\mu)^*\mathbf{IC}_\lambda, (i_\mu)^*\mathbf{IC}_\mu).$$

Now let $Z = \overline{\mathrm{Gr}^\lambda_G} \cap \overline{\mathrm{Gr}^\mu_G}$ and $f \colon Z \hookrightarrow \overline{\mathrm{Gr}^\mu_G}$ be the inclusion, which is a closed embedding. Since $(i_\mu)^*\mathbf{IC}_\lambda$ is supported in Z, then there exists a complex of sheaves $\mathcal{F} \in D^b_{\mathcal{S}}(Z,k)$ such that $\mathbf{IC}_\lambda = f_!\mathcal{F}$, given by $\mathcal{F} = f^*(i_\mu)^*\mathbf{IC}_\lambda$. Since $Z \subset \overline{\mathrm{Gr}^\lambda_G} \setminus \mathrm{Gr}^\lambda_G$, then $\mathcal{F}$ is concentrated in negative perverse degrees and since $Z \subset \overline{\mathrm{Gr}^\mu_G} \setminus \mathrm{Gr}^\mu_G$, then $f^!(i_\mu)^*\mathbf{IC}_\mu = (i_\mu f)^!\mathbf{IC}_\mu$ is concentrated in positive degrees. Therefore

$$\mathrm{Hom}_{D^b_{\mathcal{S}}(Z,k)}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) = \mathrm{Hom}_{D^b_{\mathcal{S}}(Z,k)}(\mathcal{F}, f^!(i_\mu)^*\mathbf{IC}_\mu[1]) = 0.$$

Third case: $\lambda \neq \mu$ and $\mathrm{Gr}^\lambda_G \subset \overline{\mathrm{Gr}^\mu_G}$ or $\mathrm{Gr}^\mu_G \subset \overline{\mathrm{Gr}^\lambda_G}$.

This last case, is more involved and uses the following lemma, proved on [2].

Lemma 4 (Parity vanishing). *For any $\lambda \in X^\bullet(T)$, we have*

$$\mathcal{H}^n(\mathbf{IC}_\lambda) \neq 0 \quad \textit{implies} \quad n \equiv \dim(\mathrm{Gr}^\lambda_G) \pmod 2.$$

The Verdier duality fix the simples objects $\mathbf{IC}_\lambda$ and $\mathbf{IC}_\mu$, therefore we may suppose

$\mathrm{Gr}_G^\mu \subset \overline{\mathrm{Gr}_G^\lambda}$. We are going to control separately the Gr_G^μ and the $\overline{\mathrm{Gr}_G^\mu} \setminus \mathrm{Gr}_G^\mu$ parts. Consider the inclusion $j_\mu\colon \mathrm{Gr}_G^\mu \to \mathrm{Gr}_G$ and the sheaf $(j_\mu)_*(j_\mu)^*\mathbf{IC}_\mu = (j_\mu)_*\underline{k}_{\mathrm{Gr}_G^\mu}[\dim \mathrm{Gr}_G^\mu]$. By definition, this sheaf is concentrated in perverse nonnegative degrees.

Furthermore, let $\mathscr{G}$ be the cone of $\mathbf{IC}_\mu \to (j_\mu)_*\underline{k}_{\mathrm{Gr}_G^\mu}[\dim \mathrm{Gr}_G^\mu]$. By definition, the map $\mathbf{IC}_\mu \to {}^p\mathscr{H}^0((j_\mu)_*\underline{k}_{\mathrm{Gr}_G^\mu}[\dim \mathrm{Gr}_G^\mu])$ is injective, and therefore $\mathscr{G}$ is also concentrated in perverse nonnegative degrees.

From the exact triangle

$$\mathbf{IC}_\mu \to (j_\mu)_*\underline{k}_{\mathrm{Gr}_G^\mu}[\dim \mathrm{Gr}_G^\mu] \to \mathscr{G} \to \mathbf{IC}_\mu[1]$$

we get the exact sequence

$$\begin{aligned}\mathrm{Hom}_{D^b_{\mathcal{S}}(\mathrm{Gr}_G,k)}(\mathbf{IC}_\lambda, \mathscr{G}) &\to \mathrm{Hom}_{D^b_{\mathcal{S}}(\mathrm{Gr}_G,k)}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) \\ &\to \mathrm{Hom}_{D^b_{\mathcal{S}}(\mathrm{Gr}_G,k)}(\mathbf{IC}_\lambda, (j_\mu)_*\underline{k}_{\mathrm{Gr}_G^\mu}[\dim \mathrm{Gr}_G^\mu + 1]).\end{aligned}$$

Analogously to the second case, we have the support of $\mathscr{G}$ is contained in $\overline{\mathrm{Gr}_G^\mu} \subset \overline{\mathrm{Gr}_G^\lambda} \setminus \mathrm{Gr}_G^\lambda$. If $i_\lambda : \overline{\mathrm{Gr}_G^\lambda} \setminus \mathrm{Gr}_G^\lambda \to \mathrm{Gr}_G$ is the inclusion, then $\mathscr{G} \cong (i_\lambda)_*(i_\lambda)^*\mathscr{G}$, therefore

$$\mathrm{Hom}_{D^b_{\mathcal{S}}(\mathrm{Gr}_G,k)}(\mathbf{IC}_\lambda, \mathscr{G}) = \mathrm{Hom}_{D^b_{\mathcal{S}}(\overline{\mathrm{Gr}_G^\lambda} \setminus \mathrm{Gr}_G^\lambda,k)}((i_\lambda)^*\mathbf{IC}_\lambda, (i_\lambda)^*\mathscr{G}) = 0,$$

and the first Hom is zero.

Now let's study $(j_\mu)^*\mathbf{IC}_\lambda$. Since $\mathrm{Gr}_G^\mu \subset \overline{\mathrm{Gr}_G^\lambda} \setminus \mathrm{Gr}_G^\lambda$, then $(j_\mu)^*\mathbf{IC}_\lambda$ is concentrated in degrees $< -\dim \mathrm{Gr}_G^\mu$. On the other hand, since they are in the same connected component, $\dim \mathrm{Gr}_G^\mu \equiv \dim \mathrm{Gr}_G^\lambda \pmod 2$ and, by the parity vanishing, $\mathscr{H}^{-\dim \mathrm{Gr}_G^\mu - 1}(\mathbf{IC}_\lambda) = 0$. Therefore, $\mathscr{H}^{-\dim \mathrm{Gr}_G^\mu - 1}((j_\mu)^*\mathbf{IC}_\lambda) = 0$ and $(j_\mu)^*\mathbf{IC}_\lambda$ is concentrated in degrees $\leq -\dim \mathrm{Gr}_G^\lambda - 2$, which implies

$$\mathrm{Hom}_{D^b_{\mathcal{S}}(\mathrm{Gr}_G,k)}((j_\mu)^*\mathbf{IC}_\lambda, \underline{k}_{\mathrm{Gr}_G^\mu}[\dim \mathrm{Gr}_G^\mu + 1]) = 0,$$

and with this we conclude the proof. □

The semissimplicity gives a slick proof of the equivalence stated in the introduction of the chapter (for the positive characteristic, this is still true, but the proof is harder).

Corollary 2. *The forgetful functor*

$$\mathrm{P}_{L^+G}(\mathrm{Gr}_G, k) \to \mathrm{P}_{\mathcal{S}}(\mathrm{Gr}_G, k)$$

is an equivalence of categories.

Proof. Both of these categories are semisimple and the forgetful functor send the simple objects of $\mathrm{P}_{L^+G}(\mathrm{Gr}_G, k)$ onto the simple objects of $\mathrm{P}_{\mathcal{S}}(\mathrm{Gr}_G, k)$, therefore it is an equivalence. □

4.2 The Classical Convolution

For $h \in LG$, we denote $[h] \in \mathrm{Gr}_G$ its coset and for $g \in \mathrm{Gr}_G$ and $[h] \in LG$, we denote $[g,h]$ its orbit in $\mathrm{Gr}_G \times^{L^+G} \mathrm{Gr}_G$. We have the following diagram

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{p} LG \times \mathrm{Gr}_G \xrightarrow{q} LG \times^{L^+G} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G,$$

given by $p(g,[h]) = ([g],[h])$, $q(g,[h]) = [g,h]$ and $m([g,h]) = gh$.

Since the action of L^+G in $LG \times \mathrm{Gr}_G$ is free, it follows [6] that q^* induces an isomorphism

$$D^b_{L^+G}(LG \times^{L^+G} \mathrm{Gr}_G) \xrightarrow{\sim} D^b_{L^+G \times L^+G}(LG \times \mathrm{Gr}_G).$$

For $\mathscr{F}$ and $\mathscr{G}$ in $D^b_{c,L^+G}(\mathrm{Gr}_G)$, the external tensor product is constructible equivariant $\mathscr{F} \boxtimes \mathscr{G} \in D_{c,L^+G\times L^+G}(\mathrm{Gr}_G \times \mathrm{Gr}_G)$, and there is a unique $\mathscr{F} \mathbin{\tilde{\boxtimes}} \mathscr{G} \in D^b_{c,L+G}(LG \times^{L^+G} \mathrm{Gr}_F)$ such that

$$p^*(\mathscr{F} \boxtimes \mathscr{G}) = q^*(\mathscr{F} \mathbin{\tilde{\boxtimes}} \mathscr{G}).$$

We finally define

$$\mathscr{F} \star \mathscr{G} := m_*(\mathscr{F} \mathbin{\tilde{\boxtimes}} \mathscr{G}) \in D^b_{c,L^+G}(\mathrm{Gr}_G).$$

Now let $\mathscr{F}$ and $\mathscr{G}$ be (equivariant) perverse sheaves. Is $\mathscr{F} \star \mathscr{G}$ a perverse sheaf? We have to carry our stratification throughout the process and to ensure cohomological properties.

A stratification of $LG \times^{L^+G} \mathrm{Gr}_G$ is given by

$$\widetilde{\mathrm{Gr}}_G^{\lambda,\mu} := q(p^{-1}(\mathrm{Gr}_G^\lambda \times \mathrm{Gr}_G^\mu)).$$

By definition, $\mathscr{F} \mathbin{\tilde{\boxtimes}} \mathscr{G}$ is constructible with respect to this stratification. Moreover, it is also perverse. Therefore, our questions reduces to the following: If $\mathscr{H} \in \mathrm{P}_{L^+G}(LG \times^{L^+G} \mathrm{Gr}_G)$, is it true that $m_*(\mathscr{H}) \in \mathrm{P}_{L^+G}(\mathrm{Gr}_G)$? We already saw this is true if m is stratified semi-small and locally trivial, and this is what we are going to prove.

Theorem 8. *If $\mathscr{F}$ and $\mathscr{G}$ are in $\mathrm{P}_{L^+G}(\mathrm{Gr}_G)$, then $\mathscr{F} \star \mathscr{G}$ is in $\mathrm{P}_{L^+G}(\mathrm{Gr}_G)$.*

Proof. We are going to study the map

$$m \colon LG \times^{L^+G} \mathrm{Gr}_G \to \mathrm{Gr}_G\,.$$

It clearly preserves the stratification. Moreover, the product may be restricted to a proper map $\overline{\widetilde{\mathrm{Gr}}_G^{\lambda,\mu}} \to \mathrm{Gr}_G^{\leq\lambda+\mu}$, hence m is ind-proper. Moreover, as we have argued above, $\mathscr{F} \mathbin{\tilde{\boxtimes}} \mathscr{G} \in D^b_{c,L^+G}(\mathrm{Gr}_G, k)$. Therefore, to prove $\mathscr{F} \star \mathscr{G}$ is perverse, it suffices to prove m is stratified semi-small and that m is Zariski locally trivial. Since $\dim(\mathrm{Gr}_G^\mu) = \langle 2\rho, \mu\rangle$ for $\mu \in \mathrm{X}_\bullet(T)^+$, the semi-smallness is equivalent to

$$\dim(\widetilde{\mathrm{Gr}}_G^{\lambda,\mu} \cap m^{-1}(L_\nu)) \leq \langle \rho, \lambda + \mu + \nu\rangle,$$

which is the content of the next lemma. Moreover, as we are going to see in the proof of the next lemma, we have a triangle

$$\begin{tikzcd} LG \times^{L^+G} \mathrm{Gr}_G \arrow[rr, "\sim"] \arrow[dr, "m"'] & & \mathrm{Gr}_G \times \mathrm{Gr}_G \arrow[dl, "\pi_2"] \\ & \mathrm{Gr}_G & \end{tikzcd}$$

which means the multiplication is Zariski trivial. □

Lemma 5. *For any $\lambda, \mu \in X_\bullet(T)^+$ and $\nu \in -X_\bullet(T)^+$, we have*

$$\dim(\widetilde{\mathrm{Gr}}_G^{\lambda,\mu} \cap m^{-1}(L_\nu)) \leq \langle \rho, \lambda + \mu + \nu \rangle.$$

Proof. We are going to use a result which is going to be proved in the next chapter. As we have already noticed, dimension is an important aspect in out analysis, and it is done in the finite parts of the Grassmannian given by $\mathrm{Gr}_G^{\leq\lambda}$.

Corollary 3. *Let $\lambda \in X_\bullet(T)^+$ and let $X \subset \mathrm{Gr}_G^{\leq\lambda}$ be a T-invariant closed subvariety. Then*

$$\dim X \leq \max_{\substack{\mu \in X_\bullet(T) \\ L_\mu \in X}} \langle \rho, \lambda + \mu \rangle.$$

Consider the map

$$\phi \colon LG \times^{L^+G} \mathrm{Gr}_G \to \mathrm{Gr}_G \times \mathrm{Gr}_G$$

given by $[g, h] \mapsto ([g], [gh])$. If T acts on $LG \times^{L^+G} \mathrm{Gr}_G$ by left multiplication on LG and acts on $\mathrm{Gr}_G \times \mathrm{Gr}_G$ by diagonal action, then ϕ is a T-equivariant isomorphism. Moreover, $\phi([t^\alpha, t^\beta]) = (L_\alpha, L_{\alpha+\beta})$.

Furthermore, $[t^\alpha, t^\beta]$ belongs to

$$X_{\lambda,\mu} := \overline{\widetilde{\mathrm{Gr}}_G^{\lambda,\mu}} = q(p^{-1}(\mathrm{Gr}_G^{\leq\lambda} \times \mathrm{Gr}_G^{\leq\mu}))$$

if and only if the dominant W-conjugate α^+ of α is $\leq \lambda$ and the W-conjugate β^+ of β is $\leq \mu$. Under this isomophism, $X_{\lambda,\mu} \cap m^{-1}(L_\nu)$ is sent into $\mathrm{Gr}_G^{\leq\lambda} \times \{L_\nu\}$, and therefore may be seen as a T-invariant subvariety of $\mathrm{Gr}_G^{\leq\lambda}$. Therefore

$$\dim(X_{\lambda,\mu} \cap m^{-1}(L_\nu)) \leq \max_{\substack{\alpha,\beta \in X_\bullet(T) \\ [t^\alpha, t^\beta] \in X_{\lambda,\mu}}} \langle \rho, \lambda + \alpha \rangle.$$

Finally, all of these pairs satisfy $\alpha + \beta = \nu$, therefore

$$\langle \rho, \lambda + \alpha \rangle = \langle \rho, \lambda + \nu - \beta \rangle \leq \langle \rho, \lambda + \mu + \nu \rangle,$$

as we desired to prove. □

This is going to be our monoidal structure on $\mathrm{P}_{L^+G}(\mathrm{Gr}_G)$. This is enough to construct, without difficulties, the associativity constraint.

In fact, one may analogously define $m_3 \colon LG \times^{L^+G} LG \times^{L^+G} \mathrm{Gr}_G \to \mathrm{Gr}_G$ by $m([g_1, g_2, g_3]) = g_1 g_2 g_3$ and

$$\mathrm{Conv}_3(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) := m_*(\mathcal{F}_1 \mathbin{\widetilde{\boxtimes}} \mathcal{F}_2 \mathbin{\widetilde{\boxtimes}} \mathcal{F}_3).$$

We may then use base change to find natural isomorphisms

$$(\mathscr{F}_1 \star \mathscr{F}_2) \star \mathscr{F}_3 \xleftarrow{\sim} \mathrm{Conv}_3(\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3) \xrightarrow{\sim} \mathscr{F}_1 \star (\mathscr{F}_2 \star \mathscr{F}_3).$$

To find the commutativity constraint, we must globalize the convolution!

4.3 Beilinson-Drinfeld Convolution

Remember the definition of the Grassmannian as a solution of a moduli problem. Using Beauville-Laszlo theorem, we were able to define $\mathrm{Gr}_{G,x}$ for $x \in X$ a closed point and X a smooth curve over $\mathbb{C}$. We would like to translate to maps from the definition of convolution to a moduli problem, i.e. we are going to define p, q and m using the moduli language.

$$\mathrm{Gr}_{G,x} \times \mathrm{Gr}_{G,x} \xleftarrow{p} L_x G \times \mathrm{Gr}_{G,x} \xrightarrow{q} L_x G \times^{L_x^+ G} \mathrm{Gr}_{G,x} \xrightarrow{m} \mathrm{Gr}_{G,x}$$

First of all, $\mathrm{Gr}_{G,x}$ is defined as the functor

$$A \mapsto \left\{ (\mathcal{E}, \beta) \,\middle|\, \begin{array}{l} \mathcal{E} \text{ is a } G\text{-torsor on } X_A, \text{ and} \\ \beta : \mathcal{E}|_{X_A^\times} \cong \mathcal{E}_0|_{X_A^\times} \text{ is a trivialisation} \end{array} \right\} \Big/ \text{iso}$$

and $L_x G$ is defined as the functor

$$A \mapsto \left\{ (\mathcal{E}, \alpha, \beta) \,\middle|\, \begin{array}{l} \mathcal{E} \text{ is a } G\text{-torsor on } X_A, \\ \alpha : \mathcal{E}|_{\mathbb{D}_{x,A}} \cong \mathcal{E}_0|_{\mathbb{D}_{x,A}} \text{ is a trivialisation,} \\ \beta : \mathcal{E}|_{X_A^\times} \cong \mathcal{E}_0|_{X_A^\times} \text{ is a trivialisation} \end{array} \right\} \Big/ \text{iso.}$$

Moreover, $L_x G \times^{L_x^+ G} \mathrm{Gr}_{G,x}$ represents the functor

$$A \mapsto \left\{ (\mathcal{E}_1, \mathcal{E}, \beta_1, \gamma) \,\middle|\, \begin{array}{l} \mathcal{E}_1, \mathcal{E} \text{ are } G\text{-torsor on } X_A, \\ \beta_1 : \mathcal{E}_1|_{X_A^\times} \cong \mathcal{E}_0|_{X_A^\times} \text{ is a trivialisation,} \\ \gamma : \mathcal{E}_1|_{\mathbb{D}_{x,A}} \cong \mathcal{E}|_{\mathbb{D}_{x,A}} \text{ is an isomorphism} \end{array} \right\} \Big/ \text{iso.}$$

With these definitions, the multiplication becomes a composition and p becomes a projection

$$m(\mathcal{E}_1, \mathcal{E}, \beta_1, \gamma) = (\mathcal{E}, \gamma \circ \beta_1)$$
$$p(\mathcal{E}_1, \alpha_1, \beta_1, \mathcal{E}_2, \beta_2) = (\mathcal{E}_1, \beta_1, \mathcal{E}_2, \beta_2).$$

Moreover, q becomes the functor which associates to each $(\mathcal{E}_1, \alpha_1, \beta_1, \mathcal{E}_2, \beta_2)$ the tuple $(\mathcal{E}_1, \mathcal{E}, \beta_1, \gamma)$ where $\mathcal{E}$ is obtained by gluing the bundles $\mathcal{E}_1|_{X_A^\times}$ and $\mathcal{E}_2|_{\mathbb{D}_{x,A}}$ along the isomorphism

$$\mathcal{E}_1|_{\mathbb{D}_{x,A}^\times} \xleftarrow{\beta_1} \mathcal{E}_0|_{\mathbb{D}_{x,A}^\times} \xrightarrow{\beta_2} \mathcal{E}_2|_{\mathbb{D}_{x,A}^\times},$$

which is well defined by the Beauville-Laszlo theorem, and γ is defined as the isomorphism given by this gluing process.

To globalize this whole diagram, we first remember the definition of the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{G,X}$ as the functor

$$A \mapsto \left\{ (\mathcal{E}, \beta, x) \;\middle|\; \begin{array}{l} x \in X(A), \\ \mathcal{E} \text{ is a } G\text{-torsor on } X_A, \text{ and} \\ \beta : \mathcal{E}|_{X_A \setminus \{x\}} \cong \mathcal{E}_0|_{X_A \setminus \{x\}} \text{ is a trivialisation} \end{array} \right\} \Big/ \text{iso},$$

which is an ind-scheme over X. We can define, more generally, Gr_{G,X^n}, but we are going to do only for $n = 2$, which suffices for our applications, and the generalization is immediate. The functor Gr_{G,X^2} is defined as

$$A \mapsto \left\{ (\mathcal{E}, \beta, x_1, x_2) \;\middle|\; \begin{array}{l} (x_1, x_2) \in X^2(A), \\ \mathcal{E} \text{ is a } G\text{-torsor on } X_A, \text{ and} \\ \beta : \mathcal{E}|_{X_A \setminus x_1 \cup x_2} \cong \mathcal{E}_0|_{X_A \setminus x_1 \cup x_2} \text{ trivialisation} \end{array} \right\} \Big/ \text{iso},$$

which is a ind-scheme over X^2. If $x_1 = x_2 = x$, then $\mathcal{E}|_{X_A \setminus x_1 \cup x_2} = \mathcal{E}|_{X_A \setminus \{x\}}$, therefore $\mathrm{Gr}_{G,X^2} \times_{X^2} \Delta_X \cong \mathrm{Gr}_{G,X}$. Moreover, if $x_1 \neq x_2$, we have essentially two Grassmannians, which determines an isomorphism

$$\mathrm{Gr}_{G,X^2}\big|_{X^2 \setminus \Delta_X} \cong (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_{X^2 \setminus \Delta_X},$$

given by $(\mathcal{E}, \beta, x_1, x_2) \mapsto (\mathcal{E}_1, \beta_1, x_1, \mathcal{E}_2, \beta_2, x_2)$, where $\mathcal{E}_i$ is given by gluing $\mathcal{E}|_{X_A \setminus \{x_j\}}$ and $\mathcal{E}_0|_{X_A \setminus \{x_i\}}$ along

$$\beta : \mathcal{E}|_{X_A \setminus x_1 \cup x_2} \longrightarrow \mathcal{E}_0|_{X_A \setminus x_1 \cup x_2},$$

whose inverse is defined analogously.

Remark 6. This strange behavior only appears in infinity-dimensional varieties, since in the finite case the dimension of the fiber does not grows under specialization, it can only get smaller. If we look at the varieties $\mathrm{Gr}_G^{\leq \lambda}$, then this problem is solved. In fact [26, Proposition 3.1.14], if we look at $(\mathrm{Gr}_G^{\leq \lambda} \times \mathrm{Gr}_G^{\leq \mu})\big|_{X \setminus \Delta_X}$ inside $\mathrm{Gr}_{G,X^2}\big|_{X \setminus \Delta_X}$ and denote by $\mathrm{Gr}_G^{(\lambda,\mu)}$ its closure, then

$$\mathrm{Gr}_G^{(\lambda,\mu)}\Big|_{\Delta_X} = \mathrm{Gr}_{G,X}^{\leq \lambda + \mu}.$$

We must still define the other two globalizations. First of all, the analogue of $L_x G \times \mathrm{Gr}_{G,x}$ is given by the functor

$$A \mapsto \left\{ (\mathcal{E}_1, \alpha_1, \beta_1, \mathcal{E}_2, \beta_2, x_1, x_2) \;\middle|\; \begin{array}{l} (x_1, x_2) \in X^2(A), \\ \mathcal{E}_1, \mathcal{E}_2 \text{ are } G\text{-torsor on } X_A, \\ \beta_i : \mathcal{E}_i|_{X_A \setminus \{x_i\}} \cong \mathcal{E}_0|_{X_A \setminus \{x_i\}} \text{ trivialisations,} \\ \alpha_1 : \mathcal{E}_1|_{\mathbb{D}_{x_2,A}} \cong \mathcal{E}_0|_{\mathbb{D}_{x_2,A}} \text{ trivialisation} \end{array} \right\} \Big/ \text{iso}.$$

and is denoted by $\mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X}$. The analogue of $L_x G \times^{L_x^+ G} \mathrm{Gr}_{G,x}$ is given by

$$A \mapsto \left\{ (\mathcal{E}_1, \beta_1, \mathcal{E}, \gamma, x_1, x_2) \;\middle|\; \begin{array}{l} (x_1, x_2) \in X^2(A), \\ \mathcal{E}_1, \mathcal{E} \text{ are } G\text{-torsor on } X_A, \\ \beta_1 : \mathcal{E}_i|_{X_A \setminus \{x_1\}} \cong \mathcal{E}_0|_{X_A \setminus \{x_1\}} \text{ trivialisation,} \\ \gamma : \mathcal{E}_1|_{X_A \setminus \{x_2\}} \cong \mathcal{E}|_{X_A \setminus \{x_2\}} \text{ isomorphism} \end{array} \right\} \Big/ \text{iso}.$$

and is denoted by $\mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X}$. In this way, the globalization of the initial diagram is given by

$$\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X} \xleftarrow{p} \widetilde{\mathrm{Gr}_{G,X} \times \mathrm{Gr}}_{G,X} \xrightarrow{q} \mathrm{Gr}_{G,X} \widetilde{\times} \mathrm{Gr}_{G,X} \xrightarrow{m} \mathrm{Gr}_{G,X^2},$$

where m and p are once again given by composition and projection, respectively:

$$\begin{aligned} m(\mathcal{E}_1, \mathcal{E}, \beta_1, \gamma, x_1, x_2) &= (\mathcal{E}, \gamma \circ \beta_1, x_1, x_2) \\ p(\mathcal{E}_1, \alpha_1, \beta_1, \mathcal{E}_2, \beta_2, x_1, x_2) &= (\mathcal{E}_1, \beta_1, x_1, \mathcal{E}_2, \beta_2, x_2), \end{aligned}$$

and once again, q is given by $(\mathcal{E}_1, \alpha_1, \beta_1, \mathcal{E}_2, \beta_2, x_1, x_2) \mapsto (\mathcal{E}_1, \beta_1, \mathcal{E}, \gamma, x_1, x_2)$, where $\mathcal{E}$ is given by gluing the functors $\mathcal{E}_1|_{X_A \setminus \{x_2\}}$ and $\mathcal{E}_2|_{\mathbb{D}_{x_2,R}}$ along the isomorphism

$$\mathcal{E}_1|_{\mathbb{D}^\times_{x_2,R}} \xleftarrow{\alpha_1} \mathcal{E}_0|_{\mathbb{D}^\times_{x_2,R}} \xrightarrow{\beta_2} \mathcal{E}_2|_{\mathbb{D}^\times_{x_2,R}}.$$

This gluing is not exactly justified by Beauville-Laszlo theorem, since the graph of x_2 may be bigger than a point, but this more general version is done in [4, Section 2.12]. Moreover, in the same way we had the local p and q begin L^+G-torsors, these new p and q are also torsors for the group scheme $L^+_X G$ over X defined by

$$A \mapsto \left\{ (\alpha, x) \,\middle|\, \begin{array}{l} x \in X(A), \\ \beta : \mathcal{E}_0|_{\mathbb{D}_{x,A}} \cong \mathcal{E}_0|_{\mathbb{D}_{x,A}} \text{ a trivialisation/isomorphism} \end{array} \right\}$$

(where we consider only the trivial torsor $\mathcal{E}_0$, and therefore we ignore the "up to isomorphism").

To define the convolution, we copy our previous procedure. The functors p^* and q^* are going to be once again equivalences (because they are torsors), therefore for any $\mathscr{A}, \mathscr{B} \in \mathrm{P}_{L^+_X G}(\mathrm{Gr}_{G,X}, k)$, there exists a unique $\mathscr{A} \mathbin{\widetilde{\boxtimes}} \mathscr{B}$ such that

$$q^*(\mathscr{A} \mathbin{\widetilde{\boxtimes}} \mathscr{B}) \cong p^*(\mathscr{A} \boxtimes \mathscr{B}),$$

and we define

$$\mathscr{A} \star_X \mathscr{B} := m_*(\mathscr{A} \mathbin{\widetilde{\boxtimes}} \mathscr{B}).$$

Finally, to recover the local information, we are going to take $X = \mathbb{A}^1$ and conclude the commutativity. In this case, we have global coordinates and it follows that $\mathrm{Gr}_{G,X} \cong \mathrm{Gr}_G \times X$. Moreover, define $U := X^2 \setminus \Delta_X$.

Our objective is to understand the following diagram:

$$\mathrm{Gr}_{G,X} \xhookrightarrow{i} \mathrm{Gr}_{G,X^2} \xhookleftarrow{j} (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_U$$

It breaks the Beilinson-Drinfeld Grassmannian Gr_{G,X^2} using the diagonal map. Moreover, the fact that $X = \mathbb{A}^1$ allows us to return to the Affine Grassmannian Gr_G.

Let $\pi\colon \mathrm{Gr}_{G,X} \to \mathrm{Gr}_G$ be the projection and define $\pi^\circ := \pi^*[1] = \pi^![-1]$, which takes perverse sheaves into perverse sheaves. Moreover, consider the inclusion $i\colon \mathrm{Gr}_{G,X} = \mathrm{Gr}_{G,X^2}\big|_{\Delta_X} \hookrightarrow \mathrm{Gr}_{G,X^2}$ and define $i^\circ := i^*[-1]$ and $i^\bullet := i^![1]$ and consider the inclusion $j\colon (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})\big|_U \cong \mathrm{Gr}_{G,X^2}\big|_U \hookrightarrow \mathrm{Gr}_{G,X^2}$.

The next lemma says that the convolution $\star$ in $\mathrm{P}_{L^+G(\mathrm{Gr}_G,k)}$ and the convolution $\star_X$ in $\mathrm{P}_{L^+_X G}(\mathrm{Gr}_{G,X}, k)$ are essentially the same operation. These two lemmas describe the structure of the convolution $\star_X$ in and outside the diagonal.

Lemma 6. *If $\mathscr{A}_1, \mathscr{A}_2 \in \mathrm{P}_{L^+G}(\mathrm{Gr}_G, k)$, then there are canonical isomorphisms*

$$i^\circ(\pi^\circ(\mathscr{A}_1) \star_X \pi^\circ(\mathscr{A}_2)) \cong \pi^\circ(\mathscr{A}_1 \star \mathscr{A}_2) \cong i^\bullet(\pi^\circ(\mathscr{A}_1) \star_X \pi^\circ(\mathscr{A}_2)).$$

Lemma 7. *If $\mathscr{A}_1, \mathscr{A}_2 \in \mathrm{P}_{L^+G}(\mathrm{Gr}_G, k)$, then there is a canonical isomorphism*

$$j_{!*}((\pi^\circ\mathscr{A}_1 \boxtimes \pi^\circ\mathscr{A}_2)|_U) \cong (\pi^\circ\mathscr{A}_1) \star_X (\pi^\circ\mathscr{A}_2).$$

Combining these two lemmas, we conclude a canonical isomorphism

$$\pi^\circ(\mathscr{A}_1 \star \mathscr{A}_2) \cong i^\circ(j_{!*}((\pi^\circ\mathscr{A}_1 \boxtimes \pi^\circ\mathscr{A}_2)|_U).$$

This isomorphism descrives the convolution as a particular case of a *fusion product*, defined by $i^*(j_{!*}(- \boxtimes -))[-1]$, which is more geometric. In Gr_{G,X^2}, we have a natural convolution which enables us to define the commutativity constraints.

In fact, if $\tau\colon \mathrm{Gr}_{G,X^2} \to \mathrm{Gr}_{G,X^2}$ is the automorphism that swaps x_1 and x_2, then $\tau \circ i = i$ and it stabilizes $\mathrm{Gr}_{G,X^2}\big|_U$. More precisely, under the identification $\mathrm{Gr}_{G,X^2}\big|_U \cong (\mathrm{Gr}\, G, X \times \mathrm{Gr}_{G,X})|_U$, it corresponds to the automorphism τ_U that swaps the factors $\mathrm{Gr}_{G,X}$. Therefore,

$$\begin{aligned}
\pi^\circ(\mathscr{A}_1 \star \mathscr{A}_2) &\cong i^\circ j_{!*}((\pi^\circ\mathscr{A}_1 \boxtimes \pi^\circ\mathscr{A}_2)|_U) \\
&\cong i^\circ \tau^* j_{!*}((\pi^\circ\mathscr{A}_1 \boxtimes \pi^\circ\mathscr{A}_2)|_U) \\
&\cong i^\circ j_{!*} \tau_U^*((\pi^\circ\mathscr{A}_1 \boxtimes \pi^\circ\mathscr{A}_2)|_U) \\
&\cong i^\circ j_{!*}((\pi^\circ\mathscr{A}_2 \boxtimes \pi^\circ\mathscr{A}_1)|_U) \\
&\cong \pi^\circ(\mathscr{A}_2 \star \mathscr{A}_1)
\end{aligned}$$

and we conclude the commutativity choosing an arbitrary point.

Chapter 5

The Fiber Functor

Now that we have defined the symmetric monoidal structure, let's look at the fiber functor. It's going to be

$$H^\bullet(\mathrm{Gr}_G, -)\colon \mathrm{P}_{L^+G}(\mathrm{Gr}_G) \to k\text{-}\mathsf{vect},$$

but there is a long way to go before proving its nice properties.

First of all, it might seems strange we are forgetting the graded structure of the cohomology, but we'll see that the graduation is trivial. We are going to break it into simpler parts and then we are going to sum them.

5.1 Dimension estimates

We have already seen that bounding dimension of the strata and its subvarieties is important when working with perverse sheaves. This is the subject of our next theorem.

Theorem 9. *Let $\lambda, \mu \in X_\bullet(T)$ with λ dominant.*

(1) Then

$$\overline{\mathrm{Gr}_G^\lambda} \cap S_\mu \neq \varnothing \iff L_\mu \in \overline{\mathrm{Gr}_G^\lambda} \iff \mu \in \mathrm{Conv}(W\lambda) \cap \left(\lambda + \Phi^\vee\right)$$

where Conv *denotes the convex hull.*

(2) If μ satisfies the condition in (1), then the intersection $\overline{\mathrm{Gr}_G^\lambda} \cap S_\mu$ has pure dimension $\langle \rho, \lambda + \mu \rangle$.

(3) If μ satisfies the condition in (1), then $\mathrm{Gr}_G^\lambda \cap S_\mu$ is open dense in $\overline{\mathrm{Gr}_G^\lambda} \cap S_\mu$. In particular, the irreducible components of $\overline{\mathrm{Gr}_G^\lambda} \cap S_\mu$ and $\mathrm{Gr}_G^\lambda \cap S_\mu$ are in a canonical bijection.

Proof. We are going to proof each assertion separately.

(1) We have seen the definition of S_μ as a semi-infinite orbit, but it's also possible to define it as a attracting variety of L_μ. Let $\eta \in X_\bullet(T)$ be regular dominant. Then the fixed points of η are exactly $\{L_\mu \mid \mu \in X_\bullet(T)\}$. Moreover,

$$S_\mu \subseteq \{g \in \mathrm{Gr}_G \mid \eta(a) \cdot g \to L_\mu \text{ as } a \to 0\}.$$

By the Iwasawa decomposition, this inclusion is, in fact, an equality. This can be seen in the complex points of SL_2 (or, more generally, GL_n, whose dominant regular cocharacters are determined by $(k_1, k_2, \ldots, k_n)$ such that $k_1 > k_2 > \cdots > k_n$) in the following way:

There are $\mathbb{Z}$ cocharacters, determining maps

$$\begin{aligned} \mathbb{C}^\times &\longrightarrow \mathrm{SL}_2(\mathbb{C}(\!(t)\!)) \setminus \mathrm{SL}_2(\mathbb{C}[\![t]\!]) \\ a &\mapsto \begin{pmatrix} a^n & 0 \\ 0 & a^{-n} \end{pmatrix}. \end{aligned}$$

Then for each element in LN, we have the following

$$\begin{pmatrix} a^n & 0 \\ 0 & a^{-n} \end{pmatrix} \begin{pmatrix} 1 & f(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-n} & 0 \\ 0 & a^n \end{pmatrix} = \begin{pmatrix} 1 & a^{2n} f(t) \\ 0 & 1 \end{pmatrix}$$

which goes to I as $a \longrightarrow 0$. Therefore $S_\mu = LN \cdot L_\mu$ converges to L_μ. This gives the first equivalence.

For the second equivalence, first observe

$$W\lambda \subseteq \{\mu \in X_\bullet(T) \mid L_\mu \in \mathrm{Gr}_G^\lambda\}.$$

By Cartan Decomposition, this is an equality. Moreover, $L_\mu \in \overline{\mathrm{Gr}_G^\lambda}$ if and only if, the dominant W-conjugate μ^+ of μ satisfies $\mu^+ \le \lambda$, that is,

$$W\mu \subseteq \{\nu \in X_\bullet(T) \mid \nu \le \lambda\}.$$

And this is equivalent to the desired condition by [8, chap. VIII, sec. 7, exerc. 1].

(2) We are going to prove only the upper bound on the dimension, and the lower bound uses a long induction which is written in [2].

First of all, observe $\overline{\mathrm{Gr}_G^\lambda} \cap S_\mu \neq \varnothing$ only if $\mu \le \lambda$, therefore $\overline{\mathrm{Gr}_G^\lambda} \subseteq \overline{S_\mu}$. Analogously, if $w_0 \in W$ is the longest element (so that $w_0\lambda$ is the unique antidominant element of $W\lambda$), then $\overline{\mathrm{Gr}_G^\lambda} \subseteq \overline{T_{w_0\lambda}}$.

We are going to do induction of $\langle \rho, \mu - w_0\lambda \rangle$. If $\mu = w_0\lambda$, then, by Corollary 3,

$$\overline{\mathrm{Gr}_G^\lambda} \cap \overline{S_{w_0\lambda}} \subseteq \overline{T_{w_0\lambda}} \cap \overline{S_{w_0\lambda}} = \{L_{w_0\lambda}\}.$$

Let $w_0\lambda < \mu$, let C be an irreducible component of $\overline{\mathrm{Gr}_G^\lambda} \cap \overline{S_\mu}$ and let Ψ and H_μ be defined as in Theorem 5. Define D as a irreducible component of $C \cap \Psi^{-1}(H_\mu)$. Then $\dim D \ge \dim C - 1$ and D is contained in

$$C \cap \Psi^{-1}(H_\mu) \subseteq \overline{\mathrm{Gr}_G^\lambda} \cap \overline{S_\mu} \cap \Psi^{-1}(H_\mu) \subseteq \bigcup_{\nu < \mu} \left(\overline{\mathrm{Gr}_G^\lambda} \cap \overline{S_\nu} \right)$$

which implies by induction $\dim D \le \max_{\nu<\mu} \langle \rho, \lambda + \nu \rangle = \langle \rho, \lambda + \mu \rangle + 1$. Therefore $\dim C \le \langle \rho, \lambda + \mu \rangle$.

(3) Let Z be an irreducible component of $\overline{\mathrm{Gr}_G^\lambda} \cap S_\mu$. If $Z \cap \mathrm{Gr}_G^\lambda = \varnothing$, then $Z \cap \overline{\mathrm{Gr}_G^\eta}$ for some $\eta < \lambda$. But this would contradict the dimension bound of part (2). Therefore $Z \cap \mathrm{Gr}_G^\lambda \neq \varnothing$ and it is an open dense in Z. ◻

We have essentially proved the following.

Corollary 3. *Let $\lambda \in X_\bullet(T)^+$ and let $X \subset \overline{\mathrm{Gr}_G^\lambda}$ be a closed T-invariant subvariety. Then*

$$\dim X \leq \max_{\substack{\mu \in X_\bullet(T) \\ L_\mu \in X}} \langle \rho, \lambda + \mu \rangle.$$

Proof. Let $\eta \in X_\bullet(T)$ be a regular and dominant. As we have seen in the proof of the last theorem,

$$S_\mu = \{x \in \mathrm{Gr}_G \mid \eta(a) \to L_\mu \text{ when } a \to 0\}.$$

Since X is closed and T-invariant, we conclude

$$X \subset \bigcup_{\substack{\mu \in X_\bullet(T) \\ L_\mu \in X}} S_\mu,$$

and therefore

$$X \subset \bigcup_{\substack{\mu \in X_\bullet(T) \\ L_\mu \in X}} (\mathrm{Gr}_G^{\leq \lambda} \cap S_\mu).$$

Thus the dimension bound follows from the last theorem. □

There is an analogue of the previous result where B is replaced by B^- and S_μ is replaced by T_μ.

5.2 Weight Functors

The following result says that the cohomology is not exactly graded.

Proposition 11. *For each $\mathscr{A} \in \mathrm{P}_{L^+G}(\mathrm{Gr}_G, k)$, $\mu \in X_\bullet(T)$ and $i \in \mathbb{Z}$, there is a canonical isomorphism*

$$\mathrm{H}^i_{T_\mu}(\mathrm{Gr}_G, \mathscr{A}) \xrightarrow{\sim} \mathrm{H}^i_c(S_\mu, \mathscr{A})$$

and both terms vanish if $i \neq \langle 2\rho, \mu \rangle$.

Proof. First of all, this isomorphism is a consequence of Braden's hyperbolic localization theorem [9, Theorem 1], and we are going to use it to prove the desired vanishing of cohomology.

By definition of perverse sheaves, we have $\mathscr{A}|_{\mathrm{Gr}_G^\lambda} \in D^{\leq -\langle 2\rho, \lambda \rangle}(\mathrm{Gr}_G^\lambda, k)$. Moreover, by dimensionality arguments, one has $\mathrm{H}^i_c(\mathrm{Gr}_G^\lambda \cap S_\mu, k) = 0$ for $i > \langle 2\rho, \lambda + \mu \rangle$. Now consider the distinguished triangle

$$\tau_{\leq -\langle 2\rho, \lambda \rangle - 1}(\mathscr{A}) \to \mathscr{A} \to \mathscr{H}^{-\langle 2\rho, \lambda \rangle}(\mathscr{A})[\langle 2\rho, \lambda \rangle] \xrightarrow{[1]}$$

and apply the cohomological functor $\Gamma^0_c(\mathrm{Gr}_G^\lambda \cap S_\mu, -)$ which gives a long exact sequence

$$\begin{aligned} \cdots \to \mathrm{H}^i_c(\mathrm{Gr}_G^\lambda \cap S_\mu, \tau_{\leq -\langle 2\rho, \lambda \rangle - 1}\mathscr{A}) &\to \mathrm{H}^i_c(\mathrm{Gr}_G^\lambda \cap S_\mu, \mathscr{A}) \\ &\to \mathrm{H}^i_c(\mathrm{Gr}_G^\lambda \cap S_\mu, \mathscr{H}^{-\langle 2\rho, \lambda \rangle}(\mathscr{A})[\langle 2\rho, \lambda \rangle]) \to \cdots \end{aligned}$$

Since the perverse sheaf $\mathscr{A}$ is $\mathscr{S}$-contructible, then $\mathscr{H}^{-\langle 2\rho, \lambda \rangle}(\mathscr{A}) \cong \underline{k}^{c_0}$ for some integer c_0. Therefore

$$\mathrm{H}^i_c(\mathrm{Gr}_G^\lambda \cap S_\mu, \mathscr{H}^{-\langle 2\rho, \lambda \rangle}(\mathscr{A})[\langle 2\rho, \lambda \rangle]) = \mathrm{H}^{i + \langle 2\rho, \lambda \rangle}_c(\mathrm{Gr}_G^\lambda \cap S_\mu, k^{c_0}) = 0$$

if $i > \langle 2\rho, \mu\rangle$. Therefore in this case, the map

$$\mathrm{H}^i_c(\mathrm{Gr}^\lambda_G \cap S_\mu, \tau_{\leq -\langle 2\rho,\lambda\rangle -1}\mathscr{A}) \to \mathrm{H}^i_c(\mathrm{Gr}^\lambda_G \cap S_\mu, \mathscr{A})$$

is surjective. Repeating this process, one must stop, because $\mathscr{A}$ is a bounded complex, we conclude

$$\mathrm{H}^i_c(\mathrm{Gr}^\lambda_G \cap S_\mu, \mathscr{A}) = 0$$

for $i > \langle 2\rho, \mu\rangle$. To conclude the theorem, we are going to prove $H^i_c(\overline{\mathrm{Gr}^\lambda_G} \cap S_\mu, \mathscr{A}) = 0$ and the result will follows by [24, Tag 09YP, Tag 0F75]. Consider an order λ_i in the dominant cocharacters such that $\lambda_i \leq \lambda_j \implies i \leq j$, and let i_1 be the smallest integers such that λ_{i_1} and $\mathrm{Gr}^{\lambda_{i_1}}_G \cap S_\mu \neq \varnothing$. Then this intersection is a closed subset of $\overline{\mathrm{Gr}^\lambda_G} \cap S_\mu, \mathscr{A}$ and the diagram

$$\mathrm{Gr}^{\lambda_{i_1}}_G \cap S_\mu \xhookrightarrow{i} \overline{\mathrm{Gr}^\lambda_G} \cap S_\mu \xhookleftarrow{j} (\mathrm{Gr}^{\lambda_{i_1}}_G)^c \cap \overline{\mathrm{Gr}^\lambda_G} \cap S_\mu$$

gives a distinguished triangle

$$j_! j^* \mathscr{A} \to \mathscr{A} \to i_! i^* \mathscr{A} \xrightarrow{+1} .$$

Applying the cohomological functor $\Gamma_c(\overline{\mathrm{Gr}^\lambda_G} \cap S_\mu, -)$, we get the long exact sequence

$$\cdots \to \mathrm{H}^i_c(\mathrm{Gr}^{\lambda_i}_G \cap S_\mu, \mathscr{A}) \to \mathrm{H}^i_c(\overline{\mathrm{Gr}^\lambda_G} \cap S_\mu, \mathscr{A}) \to \mathrm{H}^i_c((\mathrm{Gr}^{\lambda_1}_G)^c \cap \overline{\mathrm{Gr}^\lambda_G} \cap S_\mu, \mathscr{A}) \to \cdots$$

which gives $\mathrm{H}^i_c(\overline{\mathrm{Gr}^\lambda_G} \cap S_\mu, \mathscr{A}) \cong \mathrm{H}^i_c((\mathrm{Gr}^{\lambda_1}_G)^c \cap \overline{\mathrm{Gr}^\lambda_G} \cap S_\mu, \mathscr{A})$ for $i > \langle 2\rho, \mu\rangle$. Using induction, we are done for the first part. Analogously, we are able to prove

$$\mathrm{H}^i_{T_\mu}(\mathrm{Gr}_G, \mathscr{A}) = 0$$

for $i < \langle 2\rho, \mu\rangle$, and the theorem is proved.for $i < \langle 2\rho, \mu\rangle$, and the theorem is proved. $\square$

In this way, for any $\mu \in X_\bullet(T)$ we may define a functor $F_\mu\colon \mathrm{P}_{L^+G}(\mathrm{Gr}_G, k) \to k$-vect by

$$F_\mu(\mathscr{A}) = \mathrm{H}^{\langle 2\rho,\mu\rangle}_{T_\mu}(\mathrm{Gr}_G, \mathscr{A}) \cong \mathrm{H}^{\langle 2\rho,\mu\rangle}_c(S_\mu, \mathscr{A}).$$

Since the category $\mathrm{P}_{L^+G}(\mathrm{Gr}_G, k)$ is semisimple, this is automatically exact.

5.3 Total Cohomology

Summing these functors together, we obtain

$$F := \bigoplus_{\mu \in X_\bullet(T)} F_\mu\colon \mathrm{P}_{L^+G}(\mathrm{Gr}_G) \to k\text{-vect}.$$

The surprising fact we are going to prove next is that this is exactly the total cohomology.

Theorem 10.

(1) There is a canonical isomorphism of functors $H^\bullet(\mathrm{Gr}_G, -) \cong F$.

(2) *The functor $H^\bullet(\mathrm{Gr}_G, -)$ is exact and faithful.*

Proof.

(1) We are going to prove that

$$\mathrm{H}^p(\mathrm{Gr}_G, \mathscr{A}) \cong \bigoplus_{\substack{\mu \in X_\bullet(T) \\ \langle 2\rho, \mu\rangle = p}} F_\mu(\mathscr{A})$$

which gives, in particular, a canonical isomorphism

$$\mathrm{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \cong \bigoplus_{\mu \in X_\bullet(T)} F_\mu(\mathscr{A}).$$

Since both functors are additive, we may suppose $\mathscr{A}$ is indecomposable and, in particular, the support of $\mathscr{A}$ is connected.
For $n \in \frac{1}{2}\mathbb{Z}$, let

$$Z_n := \bigsqcup_{\substack{\mu \in X_\bullet(T) \\ \langle \rho, \mu\rangle = n}} T_\mu.$$

By an analogue of Theorem 9 (1) for T_μ, it follows

$$\bigcup_{n\in\mathbb{Z}} Z_n \quad \text{and} \quad \bigcup_{n \in \frac{1}{2}+\mathbb{Z}} Z_n$$

are unions of connected components of Gr_G. Since $\operatorname{supp}\mathscr{A}$ is connected, it must be contained in one of these sets, and let's suppose it is contained in the first one. Moreover, since Z_n is the disjoint union of the T_μ contained in it, then

$$\mathrm{H}^p_{Z_n}(\mathrm{Gr}_G, \mathscr{A}) = \begin{cases} 0, & \text{if } p \neq 2n, \\ \bigoplus_{\langle \rho, \mu\rangle = n} F_\mu(\mathscr{A}), & \text{if } p = 2n. \end{cases}$$

Moreover, by an analogue of Proposition 5 for T_μ, we get

$$\overline{Z_n} = Z_n \sqcup Z_{n+1} \sqcup Z_{n+2} \sqcup \cdots = Z_n \sqcup \overline{Z_{n+1}}.$$

Therefore, for the diagram

$$Z_n \overset{j}{\hookrightarrow} \overline{Z_n} \overset{i}{\hookleftarrow} \overline{Z_{n+1}}$$

we have an associated distinguished triangle

$$i_* i^! \mathscr{A}_n \longrightarrow \mathscr{A}_n \longrightarrow j_* j^! \mathscr{A}_n \longrightarrow i_* i^! \mathscr{A}_n[1]$$

where $\mathscr{A}_n$ is the corestriction of $\mathscr{A}$ to $\overline{Z_n}$. Applying the cohomological functor $\mathrm{H}^0(\overline{Z_n}, -)$ we get the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^p_{\overline{Z_{n+1}}}(\mathrm{Gr}_G, \mathscr{A}) \longrightarrow \mathrm{H}^p_{\overline{Z_n}}(\mathrm{Gr}_G, \mathscr{A}) \longrightarrow \mathrm{H}^p_{Z_n}(\mathrm{Gr}_G, \mathscr{A}) \longrightarrow \cdots.$$

Moreover $\mathscr{A}$ is a finite sum of $\mathbf{IC}_{\lambda_i}$ for some λ_i, and each one of these have compact support, because it is contained in $\mathrm{Gr}_G^{\leq\lambda}$. Therefore, for for n big enough $\overline{Z_n} \cap \operatorname{supp}\mathscr{A} = \varnothing$, therefore $\mathrm{H}^{\bullet}_{\overline{Z_n}}(\mathrm{Gr}_G, \mathscr{A}) = 0$. Using decreasing induction on n on the long exact sequence, we are able to prove

$$\mathrm{H}^{p}_{\overline{Z_n}}(\mathrm{Gr}_G, \mathscr{A}) = 0 \quad \text{if } p \text{ is odd or } n > \frac{p}{2}$$

and

$$\mathrm{H}^{p}_{\overline{Z_n}}(\mathrm{Gr}_G, \mathscr{A}) \cong \mathrm{H}^{p}_{Z_{p/2}}(\mathrm{Gr}_G, \mathscr{A}) \quad \text{if } p \text{ is even and } n \leq \frac{p}{2}.$$

Taking n small enough so that $\operatorname{supp}\mathscr{A} \subseteq \overline{Z_n}$ one concludes the proof.

(2) Since both categories are semisimple and the functor is additive, then it is also exact. To prove faithfulness, we may suppose $\mathscr{A} = \mathbf{IC}_\lambda$ is simple, once again because the category is semisimple. In this case, the support of $\mathscr{A}$ is contained in $\overline{\mathrm{Gr}_G^\lambda}$ and $\mathscr{A}|_{\mathrm{Gr}_G^\lambda} = \underline{k}[\dim \mathrm{Gr}_G^\lambda]$. Moreover

$$((\operatorname{supp}\mathscr{A}) \setminus \mathrm{Gr}_G^\lambda) \cap T_\lambda = \varnothing \quad \text{and} \quad \mathrm{Gr}_G^\lambda \cap T_\lambda = \{L_\lambda\}.$$

This implies $F_\lambda(\mathscr{A}) \neq 0$ and, in particular, $F(\mathscr{A}) \neq 0$. □

5.4 Compatibility

Now, we must prove the compatibility conditions of this fiber functor. First of all, since the functor is exact, it's enough to compute its imagem under the simple objects.

Proposition 12. *Let $\lambda, \mu \in X_\bullet(T)$ with λ dominant. Then $\dim F_\mu(\mathbf{IC}_\lambda)$ is the number of irreducible components of $\mathrm{Gr}_G^\lambda \cap S_\mu$. In particular, it's nonzero if and only if $\mu \in \mathrm{Conv}(W\lambda) \cap (\lambda + \Phi^\vee)$.*

Proof. By the constriants for the intersection cohomology, for $\eta \in X_\bullet(T)^+$, exactly one of these conditions holds:

- $\mathrm{Gr}_G^\eta \cap \operatorname{supp}\mathbf{IC}_\lambda = \varnothing$ and therefore $\mathbf{IC}_\lambda|_{\mathrm{Gr}_G^\eta} = 0$;
- $\eta = \lambda$ and therefore $\mathbf{IC}_\lambda|_{\mathrm{Gr}_G^\eta} = \underline{k}[\langle 2\rho, \lambda\rangle] \in D^{-\langle 2\rho,\lambda\rangle}(\mathrm{Gr}_G^\eta, k)$;
- $\eta < \lambda$ and therefore $\mathbf{IC}_\lambda|_{\mathrm{Gr}_G^\eta} \in D^{-\langle 2\rho,\lambda\rangle-1}(\mathrm{Gr}_G^\eta, k)$.

As in the proof of Proposition 11, we are able to reconstruct the cohomology of $\mathrm{H}^p(S_\mu, \mathbf{IC}_\lambda)$ from the intersectionwith the Schubert cells Gr_G^λ and we are able to conclude

$$\mathrm{H}_c^{\langle 2\rho,\mu\rangle}(S_\mu, \mathbf{IC}_\lambda) = \mathrm{H}_c^{\langle 2\rho,\mu\rangle}(\mathrm{Gr}_G^\lambda \cap S_\mu, \mathbf{IC}_\lambda|_{\mathrm{Gr}_G^\lambda})$$

and therefore

$$F_\mu(\mathbf{IC}_\lambda) = \mathrm{H}_c^{\langle 2\rho,\lambda+\mu\rangle}(\mathrm{Gr}_G^\lambda \cap \mu, k),$$

which is exactly the top cohomology with compact support, which has a natural basis indexed by the irreducible components of top dimension. □

Now we must relate it to the monoidal structure, which is the following result.

Theorem 11. *For any $\mathscr{A}_1, \mathscr{A}_2$ in $\mathrm{P}_{L^+G}(\mathrm{Gr}_G)$, there exists a canonical isomorphism*

$$F(\mathscr{A}_1 \star \mathscr{A}_2) \cong F(\mathscr{A}_1) \otimes F(\mathscr{A}_2).$$

The proof of this theorem and of compatibility with the constraints are given in [2]. If one work naively, then the commutativity constraint of $\mathrm{P}_{L^+G}(\mathrm{Gr}_G, k)$ is sent into the supercommutativity[1] constraint of k-vect($\mathbb{Z}$), where the grading is given by the cohomology $\mathrm{H}^\bullet$. We must, then, change our commutativity constraint from $\mathrm{P}_{L^+G}(\mathrm{Gr}_G, k)$ to apply the Tannakian formalism.

[1]If v has degree i and w has degree j, then $wv = (-1)^{ij} vw$.

Chapter 6

Finding the Group

We have defined a k-linear Abelian category $\mathrm{P}_{L^+G}(\mathrm{Gr}_G, k)$ endowed with symmetric monoidal $\star$ structure and a faithful exact symmetric monoidal k-linear functor $F\colon \mathrm{P}_{L^+G}(\mathrm{Gr}_G, k) \longrightarrow k\text{-vect}$. Moreover, to ensure the rigidity condition, it is enough [10, Proposition 1.20] to prove that if $\dim F(\mathscr{A}) = 1$ for some $\mathscr{A} \in \mathrm{P}_{L^+G}(\mathrm{Gr}_G, k)$, then invertible under the monoidal structure $\star$. But if $\mathbf{IC}_\lambda$ satisfies $\dim F(\mathbf{IC}_\lambda) = 1$, then by Proposition 11, λ must be orthogonal to every root $\mu \in \Phi$. This means $\mathrm{Gr}_G^\lambda = \{L_\lambda\}$ and its inverse is given by $\mathbf{IC}_{-\lambda}$.

Therefore, by the Tannakian formalism, there exists an affine group scheme $\widetilde{G}_k$ over k such that $\mathrm{P}_{L^+G}(\mathrm{Gr}_G, k) \cong \mathsf{Rep}_k(\widetilde{G}_k)$.

6.1 Geometric properties

First of all, we are going to use Proposition 2 to ensure geometric properties of $\widetilde{G}_k$. Later, we are going to study the torus and the root datum.

The main observation of the following two proofs is the following: if λ_1 and λ_2 are two dominant cocharaters, then $\mathbf{IC}_{\lambda_1+\lambda_2}$ is a sommand of $\mathbf{IC}_{\lambda_1} \star \mathbf{IC}_{\lambda_2}$.

Proposition 13. *The affine group scheme $\widetilde{G}_k$ is algebraic.*

Proof. Let $\lambda_1, \ldots, \lambda_n$ be dominant cocharacters generating $X_\bullet(T)^+$. By the observation above, the smallest monoidal Abelian category containing $\mathscr{A} := \mathbf{IC}_{\lambda_1} \oplus \cdots \oplus \mathbf{IC}_{\lambda_n}$ is the whole category $\mathrm{P}_{L^+G}(\mathrm{Gr}_G, k)$, which implies the group is algebraic. □

Proposition 14. *The linear algebraic group $\widetilde{G}_k$ is connected.*

Proof. Let $\mathscr{A} \in \mathrm{P}_{L^+G}(\mathrm{Gr}_G, k)$ be an arbitrary perverse sheaf and suppose $\mathbf{IC}_\lambda$ is a summand of $\mathscr{A}$ such that λ is maximal among such dominant cocharacters. Then $\mathbf{IC}_{2\lambda}$ is a subobject of $\mathscr{A} \star \mathscr{A}$, but is not a subquotient of $\mathscr{A}^n$ for any $n \geq 0$. □

Proposition 15. *The connected linear algebraic group $\widetilde{G}_k$ is reductive.*

Proof. If $\overline{k}$ is the algebraic closure of k, then we must prove the category $\mathsf{Rep}_{\overline{k}}(\widetilde{G}_{\overline{k}}) \cong \mathrm{P}_{L^+G}(\mathrm{Gr}_G, \overline{k})$ is semisimple, which was already proven in Theorem 7. Therefore $\widetilde{G}_k$ is a reductive group. □

6.2 Root datum

Let's first contruct the maximal split torus of $\widetilde{G}_k$.

By construction, the fiber functor

$$F := \bigoplus_{\mu \in X_\bullet(T)} F_\mu \colon \mathrm{P}_{L^+G}(\mathrm{Gr}_G, k) \to k\text{-vect}$$

factors through the category of $X_\bullet(T)$-graded vector spaces k-vect$(X_\bullet(T))$

$$\begin{tikzcd} \mathrm{P}_{L^+G}(\mathrm{Gr}_G, k) \arrow[r, "F'"] \arrow[rr, bend right, "F"'] & k\text{-vect}(X_\bullet(T)) \arrow[r, "U"] & k\text{-vect}. \end{tikzcd}$$

Let $T_k^\vee$ be the unique k-split torus such that $X^\bullet(T_k^\vee) = X_\bullet(T)$. Then

$$k\text{-vect}(X_\bullet(T)) \cong k\text{-vect}(X^\bullet(T_k^\vee)) \cong \mathsf{Rep}_k(T_k^\vee)$$

and F' is compatible with the monoidal structure. Therefore there exists a unique morphism [2, Proposition 2.10]

$$\varphi \colon T_k^\vee \to \widetilde{G}_k.$$

This morphism is a closed embedding by [10, Proposition 2.21], and $T_k^\vee$ is a maximal torus by [2, (9.1)].

In what follows, we are going to assume k algebraically closed. Now we must identify the root data of $(T_k^\vee, \widetilde{G}_k)$ with the Langlands dual of G. First of all, let $\widetilde{B}$ be a Borel subgroup of $\widetilde{G}_k$ containing $T_k^\vee$ such that 2ρ is a dominant cocharacter under the order induced by $\widetilde{B}$. This group is not necessarily unique by this definition, but the next result implies its uniqueness.

Proposition 16. *The dominant characters of $T_k^\vee$ are exactly the dominant cocharacters of T, i.e. $X^\bullet(T_k^\vee)^+ = X_\bullet(T)^+$.*

Proof. Let $\lambda \in X_\bullet(T)^+$ be a dominant cocharacter and let V_λ be the simple representation of $G_k^\vee$ corresponding to $\mathbf{IC}_\lambda$. Then by Proposition 11, the weights of V all satisfy $\mu \leq \lambda$, and therefore λ is the maximal element of this representation, being the highest weight and a dominant character.

Reciprocally, let $\mu \in X^\bullet(T_k^\vee)^+$ be a dominant character and let V_μ be the $\widetilde{G}_k$-representation of highest weight μ. If $\lambda \in X_\bullet(T)$ is defined as the cocharacter such that $\mathbf{IC}_\lambda$ corresponds to V_μ, then what we have argued above implies $\lambda = \mu$ and we have the desired correspondence. □

Remark 7. Although this is not trivial, these subgroups and elements are canonical. For example, the fiber functor $F = \bigoplus_\mu F_\mu$ depends on the choice of the torus T. But actually, there is a canonical isomorphism between these functors for two differentes tori T and T', and so $T_k^\vee$ is uniquely determined as a subgroup of $\widetilde{G}_k$.

Now let $G_k^\vee$ be the Langlands dual of G. Then $T_k^\vee$ is also a maximal torus of $G_k^\vee$ and we are essentially going to compare the pairs $(\widetilde{G}_k, T_k^\vee)$ and $(G_k^\vee, T_k^\vee)$ endowed also with their canonical choices of positive roots.

Proposition 17. *Both groups have the same subset simple roots. More precisely, $\Phi_s(\widetilde{G}_k, \widetilde{B}, T_k^\vee) = \Phi_s^\vee(G, B, T)$ as subgroups of $X^\bullet(T_k^\vee) = X_\bullet(T)$.*

Proof. Let $\lambda \in X_\bullet(T)^+$ and let V_λ be the $G_k^\vee$-representation with highest weight λ and W_λ the $\widetilde{G}_k$-representation with heighest weight λ. One again by Proposition 11, both of them has the same set of weight

$$\mathrm{Conv}(W\lambda) \cap (\lambda + \Phi^\vee) = \{\mu \in X_\bullet(T) \mid \mu - \lambda \in \Phi^\vee \text{ and } \mu \in \mathrm{Conv}(W\lambda)\}.$$

Moreover, the set

$$\{\lambda - \mu \in X_\bullet(T) \mid \mu \text{ a weight of } V_\lambda\}$$

is exactly the set of $\mathbb{N}$-combinations of the positive coroots of $G_k^\vee$. Since this set, as argued above, is the same as the $\mathbb{N}$-combinations of the positive coroots of $\widetilde{G}_k$, their irreducible elements agree, and they are exacly the simple roots. □

This final proposition will end our proof.

Proposition 18. *The group $\widetilde{G}_k$ is the Langlands dual of G. In other words, their root datum are dual to one another.*

Proof. By construction, the characters and cocharacters are already dual to one another. Moreover, we have already identified $\Phi_s(\widetilde{G}_k, \widetilde{B}, T_k^\vee) = \Phi_s^\vee(G, B, T)$. We must prove

$$\Phi_s^\vee(\widetilde{G}_k, \widetilde{B}, T_k^\vee) = \Phi_s(G, B, T)$$

and that the correspondence $\alpha \mapsto \alpha^\vee$ is the same. Observe that, since simple roots and coroots generate the roots and coroots, everything might be simply proved with them and the result will follows.

First of all, note that Proposition 16 implies

$$\{\mathbb{Q}_+ \cdot \alpha \mid \alpha \in \Phi_s^\vee(\widetilde{G}_k, \widetilde{B}, T_k^\vee)\} = \{\mathbb{Q}_+ \cdot \beta \mid \beta \in \Phi_s(G, B, T)\}.$$

In fact, both of them are the extremal rays of the set

$$\{\lambda \in \mathbb{Q} \otimes X^\bullet(T) \mid \langle \lambda, \mu \rangle \geq 0 \text{ for all } \mu \in X_\bullet(T)^+\}.$$

Now let $\alpha \in \Phi_s(G, B, T)$ be a simple coroot. As we have proved in the last proposition, $\alpha^\vee \in \Phi_s(\widetilde{G}_k)$. Let $\widetilde{\alpha} \in \Phi_s^\vee(\widetilde{G}_k, \widetilde{B}, T_k^\vee)$ be the dual simple coroot, we would like to prove $\alpha = \widetilde{\alpha}$. In fact, by what was argued above, we know $\widetilde{\alpha}$ is a $\mathbb{Q}_+$-multiple of some simple root of G. Moreover, $\widetilde{\alpha}$ satisfies, for example $\langle \widetilde{\alpha}, \alpha \rangle = 2$, which implies $\alpha = \widetilde{\alpha}$ and concludes the proof. □

Appendix A

Algebraic Groups

Let k be a field of characteristic 0. An affine group scheme G over k is a representable functor
$$G\colon k\text{-}\mathsf{Alg} \to \mathsf{Grp}.$$

When G is reduced and of finite type, we call G an algebraic group. An affine algebraic group G is also called a linear algebraic group.

Example 14. The most basic examples are the following.

(i) The functor $\mathbb{G}_a\colon A \mapsto (A, +, 0)$ is an algebraic group represented by $k[X]$;

(ii) The functor $\mathbb{G}_m\colon A \mapsto (A^\times, \cdot, 1)$ is an algebraic group represented by the k-algebra $k[X, X^{-1}]$;

(iii) The functor $\mathrm{GL}_n\colon A \mapsto \mathrm{GL}_n(A)$ is an algebraic group represented by the k-algebra $k[X_{11}, X_{12}, \ldots, X_{nn}, 1/\det(X_{ij})]$.

A *torus* T over k is a group such that $T_{\overline{k}} \cong (\mathbb{G}_m)^n$ over $\overline{k}$ for some $n \geq 0$, and the torus is k-split if $T \cong (\mathbb{G}_m)^n$ over k.

Example 15. The special orthogonal group SO_2 over $\mathbb{R}$ is a torus which is not $\mathbb{R}$-split.

Definition 15. Let T be a split torus. The *character group* is defined as $X^\bullet(T) := \mathrm{Hom}_k(T, \mathbb{G}_m)$, and the *cocharacter group* is defined as $X_\bullet(T) := \mathrm{Hom}_k(\mathbb{G}_m, T)$.

A classical computation yields $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$, which implies the composition defines a perfect pairing
$$\langle -, - \rangle\colon X^\bullet(T) \times X_\bullet(T) \to \mathbb{Z}.$$

Theorem 12. *Let G be a smooth connected linear algebraic k-group. All maximal k-split tori $T \subseteq G$ are $G(k)$-conjugate.*

Let G be a smooth connected linear algebraic k-group. We call G k-split if its maximal tori are k-split. If T is a maximal torus on G, then we call the Weyl group associated to (G, T) by
$$W(G, T) := N_G(T)/C_G(T).$$

Example 16. For $G = \mathrm{GL}_n$ or SL_n the diagonal matrices form a maximal torus called *standard torus* and they are k-split. In both of these cases, the Weyl group is S_n and the representatives in GL_n are given by the permutation matrices.

Let k be an algebraically closed field, and G an algebraic group over k. An element $g \in G(k)$ is called *semisimple* if there is a faithful representation $G \hookrightarrow \mathrm{GL}_n$ such that g is diagonalizable. If k is not algebraically closed, then we call $g \in G$ semisimple if it becomes diagonalizable in $\overline{k}$.

Definition 16. A connected algebraic group G is said to be *solvable* if there is a series

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_t = \{1\}$$

such that each quotient G_{i+1}/G_i is commutative. An algebraic group G is said to be *unipotent* if every nonzero representation of G has a nonzero fixed vector.

Let G be a connected algebraic group over k. There is the maximal connected solvable normal subgroup $R(G)$ called *radical* of G. A group is called semisimple if $R(G_{\overline{k}}) = \{1\}$. If k is algebraically closed, then $G/R(G)$ is semisimple.

Analogously, we may define $R_u(G)$ as the maximal connected unipotent normal subgroup and call it *unipotent radical* of G. A group is called reductive if $R_u(G_{\overline{k}}) = \{1\}$. If k is algebraically closed, then $G/R_u(G)$ is reductive.

Every unipotent group is solvable, therefore $R_u(G) \subseteq R(G)$. In particular, every semisimple group is reductive, but the converse is false.

Example 17. The group SL_n is semisimple, whereas the group GL_n is only reductive.

Another important subgroup of G to study its representation theory is the Borel subgroup.

Definition 17. Let G be a connected algebraic group. A *Borel subgroup* $B \subseteq G$ is a maximal closed solvable connected subgroup.

Theorem 13. *Let G be a connected algebraic group. Then all Borel subgroups are conjugated and, if B is a Borel subgroup, then G/B is projective. Moreover, $N_G(B) = B$.*

Example 18. If $G = \mathrm{GL}_n$ or SL_n, then the subgroup of upper triangular matrices is a Borel subgroup B. In this case, G/B is isomorphic to the set of *flags* $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0$, where each V_i has codimension i. Then

$$G/B \subset \mathrm{Gr}(n,1) \times \mathrm{Gr}(n,2) \times \cdots \times \mathrm{Gr}(n,n-1) \subset \mathbb{P}^N$$

for some N big enough.

This property is not reserved for the Borel subgroup.

Definition 18. Let G be a smooth connected algebraic k-group. We call $P \subsetneq G$ a parabolic subgroup if G/P is a complete variety.

Theorem 14. *A subgroup $P \subsetneq G$ is parabolic if and only if it contains a Borel subgroup. In particular, Borel subgroups are minimal parabolic subgroups.*

The parabolic subgroups containing a given Borel subgroup are going to be later classified.

The two most important representations a connected reductive algebraic group G has are the adjoint and the coadjoint representations on its Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$ and its dual $\mathfrak{g}^\vee$. Let $T \subseteq G$ be a maximal torus and $\mathfrak{h} = \mathrm{Lie}(T)$ be its Lie algebra. Since T is commutative and its elements are semisimple, then

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in X^\bullet(T)} \mathfrak{g}_\alpha .$$

A root of (G,T) is an $\alpha \in X^\bullet(T)$ such that $\mathfrak{g}_\alpha \neq 0$, and the set of roots is denoted by Φ. Analogously we define coroots $\alpha^\vee$ and denote the set of coroots by $\Phi^\vee$. If B is a Borel subgroup such that $T \subseteq B \subseteq G$, then we define the positive roots Φ_+ as the roots $\alpha \in \Phi$ such that $\mathfrak{g}_\alpha \subset \mathfrak{b} = \mathrm{Lie}(B)$. The simple roots are the positive roots which are not written as a sum of other positive roots and the set of simple roots is denoted by Φ_s. Analogously, we are also able to define the same notions for the coroots.

Example 19. Let $G = \mathrm{GL}_n$, T be the diagonal matrices and B the upper triangular matrices. Then both $X^\bullet(T)$ and $X_\bullet(T)$ are canonically isomorphic to $\mathbb{Z}^n$. The roots are given by $e_i - e_j$ for $i \neq j$, where $e_i = (0, 0, \ldots, 1, \ldots, 0)$ where 1 is placed in the i-th coordinate. The positive roots are given by $e_i - e_j$ for $i < j$. Finally, the simple roots are given by $e_i - e_{i+1}$ for $1 \leq i < n$.

For each root α, the coroot $\alpha^\vee$ satisfies $\langle \alpha, \alpha^\vee \rangle = 2$, and the automorphism

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$$

is a reflection through the hyperplane $\{x \in X^\bullet(T) \mid \langle x, \alpha^\vee \rangle = 0\}$, and such that $s_\alpha(\alpha) = -\alpha$.

It's not hard to see that the Weyl group $W(G, T)$ acts faithfully on $X^\bullet(T)$. Surprisingly, its image has a simple description.

Proposition 19. *The subgroup $W' \subset \mathrm{Aut}(X^\bullet(T))$ generated by the reflections s_α is isomorphic to $W(G, T)$.*

The choice of positive roots also determines the set of dominant cocharacters $X^\bullet(T)^+$. They are exactly the $\lambda \in X_\bullet(T)$ such that

$$\langle \alpha, \lambda \rangle \geq 0 \quad \text{for every } \alpha \in \Phi^+.$$

For $\lambda, \mu \in X^\bullet(T)^+$, we write $\mu \leq \lambda$ when $\lambda - \mu$ is a sum of positive coroots.

If we define

$$2\rho := \sum_{\alpha \in \Phi^+} \alpha,$$

then

$$\mu \leq \lambda \implies \langle \rho, \lambda - \mu \rangle \text{ is a non-negative integer.}$$

Moreover, we say a cocharacter λ is dominant if

$$\langle \alpha, \lambda \rangle \neq 0 \quad \text{for every } \alpha \in \Phi.$$

Theorem 15. *Let k be an algebraically closed field. The data $(X^\bullet(T), \Phi, X_\bullet(T), \Phi^\vee)$ is called root data and determines a unique reductive group up to isomorphism. If k is not algebraically closed, then this determines a unique k-split reductive group.*

This allows us to define the Langlands dual group of a reductive group G as the unique group $G^\vee$ whose root data is dual to the one given by G.

Finally, let B be a Borel subgroup of G and let $I \subsetneq \Phi_s$ a proper subset of the simple roots. Let W_I be the subgroup of the Weyl group generated by the reflexions s_α for $\alpha \in I$. The the subset

$$P_I := \bigsqcup_{w \in W_I} BwB$$

is a parabolic subgroup of G, where each $w \in N_G(T)$ is a representative of its class in W_I. Moreover, every parabolic subgroup of G containing B is of this form.

Example 20. If $G = \mathrm{GL}_n$, T = diagonal matrices and B = upper triangular matrices, then the parabolic subgroup containing B are exactly the subgroups of block diagonal matrices. For example, if $I = \{e_1 - e_2\}$, then P is the subgroup of the invertible matrices of the form

$$\begin{pmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{pmatrix}.$$

Appendix B

Ind-Schemes

When studying schemes, we encounter "infinite-dimensional spaces" which are too big to be schemes, but are almost schemes. For example, $\mathbb{A}^\infty$, $\mathbb{P}^\infty$ and, our most important example Gr_G. The proofs of the statements here stated are written in [22].

Definition 19. A functor $X\colon k\text{-}\mathsf{Alg} \to \mathsf{Set}$ is called an *ind-scheme* if it is isomorphic to a filtered colimit of representable functors X_i with transition morphism being closed immersions

$$X \cong \operatorname{colim} X_i.$$

The category of ind-schemes is denoted by IndSch.

This should be called an *strict* ind-scheme, since it's stronger than simply being an ind-object, but since we are working only with this definition (which seems to be the correct one when working with geometric objects), we'll ignore the "strict".

Example 21. Let $\mathbb{A}_k^\infty$ be the functor defined by $A \mapsto \bigoplus_{i=0}^\infty A$. Then, we may write it as $\mathbb{A}_k^\infty \cong \operatorname{colim} \mathbb{A}_k^n$ with the transition maps as the obvious inclusions. Since each $\mathbb{A}_k^n$ is affine, the ind-scheme $\mathbb{A}_k^\infty$ is *ind-affine*.

Example 22. Let $\mathbb{P}_k^\infty\colon \mathsf{AffSch}_k \to \mathsf{Set}$ be the functor defined by

$$T \mapsto (\mathcal{L}, (s_i)_{i=0}^\infty)/\sim,$$

where $\mathcal{L}$ is a line bundle and s_i are sections which generate $\mathcal{L}$ and which are 0 for all but finitely many. Then $\mathbb{P}_k^\infty = \operatorname{colim} \mathbb{P}_k^n$ and, since each $\mathbb{P}_k^n$ is projective (and, in particular, proper), the ind-scheme $\mathbb{P}_k^\infty$ is *ind-projective* (and, in particular, *ind-proper*).

We are not assuming any geometric hypothesis, because we began with affine schemes, and the quasi-compactness implies it is a sheaf.

Theorem 16. *Every ind-scheme X is a sheaf in the fpqc topology.*

As usual with ind-constructions, its category of objects is well-behaved. For example, it admits all finite limits and arbitrary disjoint unions. Moreover, most definitions and properties of schemes are inherited by ind-schemes.

Definition 20. The underlying topological space of an ind-scheme X is defined as

$$|X| := \operatorname{colim}_k X(k)$$

where the colimit is taken over the category of fields. Its topology is generated by subfunctor represented by open immersions.

If X is a scheme, this is the usual underlying topological space.

Lemma 8. *If* $X = \operatorname{colim}_i X_i$ *is a presentation of an ind-scheme, then the natural map* $\operatorname{colim}_i |X_i| \xrightarrow{\sim} |X|$ *is an isomorphism.*

We may define an open or closed sub-ind-scheme analogously as in the scheme-theoretic case.

Definition 21. An ind-scheme X is reduced if there is a presentation $X = \operatorname{colim}_i X_i$ such that each X_i is reduced.

Remark 8. This definition may be given for any local property. For example, being smooth, flat, unramified, etc.

For each ind-scheme X, there is a unique reduced ind-scheme $X_{\mathrm{red}} \subseteq X$ such that $X_{\mathrm{red}}(T) = X(T)$.

Most properties studied in Algebraic Geometry are only local on target. We are also able to define analogues of these properties for ind-schemes.

Definition 22. Let **P** be a property stable under base change and Zariski local on target and let X and Y be two ind-scheme. We say $f: X \to Y$ has property ind-**P** if there are presentations $X = \operatorname{colim}_i X_i$ and $Y = \operatorname{colim}_j Y_j$ such that each $f_{i,j}: X_i \to Y_j$ has property **P**.

Actually, we allow properties which are only be stable under base change with a closed immersion. Some of the properties satisfying these constraints are: proper, projective, affine, quasi-compact, finite type, etc.

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